ITEP/TH-58/97
FIAN/TD-13/97
hep-th/9711006

τ-function within group theory approach and its quantization

A.Mironov [1]

Theory Department, P.N.Lebedev Physics Institute, Moscow, Russia
and
ITEP, Moscow, Russia

Abstract

This is a review of the results related to generalizations of the notion of τ-function and integrable hierarchies and to their interpretation within the group theory framework that admits an immediate quantization procedure. Different group theory structures underlying integrable system are discussed in detail as well as their quantum deformations.

Contents

[1] E-mail address: mironov@lpi.ac.ru, mironov@itep.ru

1 Introduction

During last years, integrability gets more and more important in particle physics and string theory [1, 2]. One of the main lessons that can be learned from the numerous concrete examples is that the *classical* integrability in *quantum* field theories turns out to be surprisingly typical.

One of the first examples of using essentially integrable systems in this field is attempts to perturbatively deform conformal field theories along the "integrable" directions [3]. This is ultimately necessary for constructing the non-perturbative solutions in string theory. The perturbative deformations certainly describe only the vicinity of the critical points, while not exposing the global structure of the configuration space of string theory. Still, it is the papers [3] (see also [4]) that have already pointed out the *integrable* deformations of conformal field theories. Moreover, within the string theory approach it has been also expected that the sum of the whole perturbative series for the amplitudes (and the partition function) can be described by a quantum integrable system [5]. At this point, it is of great importance that the effective action for the *quantum* integrable system is a τ-fucntion of some *classical* integrable system[1]. This amazing phenomenon has been already realized in concrete examples [8]-[12], while never been explained.

It is interesting that all the examples can be parted into two large classes of the systems naturally depending on times (coupling constants) and on the so called Miwa variables, these latter ones being just eigenvalues of an (infinite) external matrix. An instance of the first class system is a functional of the correlators in the quantum non-linear Schrödinger model, which is a τ-function of some classical integrable non-linear equation of the same kind [9]. A typical second class example is given by the partition function in the six-vertex model with non-trivial boundary conditions. This partition function turns out to be the τ-function of the $2d$ Toda system expressed in Miwa variables (that are, in essence, the boundary conditions) [12].

However, the most important case of the integrability phenomenon under consideration is given

[1]Surprisingly enough, integrating some objects like τ-function [6] over the universal moduli space in string theory [7] leads to a τ-function given on the same universal moduli space.

by the matrix models [13, 14, 2]. Among other things, in the matrix models the both types of time variables are realized. Indeed, the partition function of the discrete matrix models is a τ-function, the times being coupling constants, while that of the continuous matrix models is a τ-function depending on an external matrix that gives rise to Miwa variables.

In fact, the classical integrability in effective field theory is quite typical and non-specific, say, for two dimensions. A good example is given by the solution of the $4d$ $N = 2$ supersymmetric Yang-Mills theory, which has been recently constructed by N.Seiberg and E.Witten [15]. Classical integrability in this case has been revealed in the paper [16] (for recent progress see [1, 17, 18] and references therein). Since integrability looks so typical, one could expect for the effective action in typical field theory to be a τ-function provided the variables are choosen properly and all possible interactions are taken into account. In practice, these latter are usually restricted by some additional requirements (renormalizability etc).

It was already noted above that, although the integrability phenomenon has been investigated in a series of examples, its origin still remains mysterious. In the present paper, we deal with issues related to the general properties of quantum and classical τ-functions. This would help us to shed some light on the phenomenon. Besides, we discuss here some immediate applications of the general approach. For example, we calculate the S-matrix of the quantum Liouville theory.

In order to discuss how correlators of quantum integrable system are related to classical hierarchies, one needs to deal with the quantum system in terms suitable for the classical hierarchies. The key notion in these hierarchies is the τ-function, while the quantum integrable systems have been formulated so far only in R-matrix terms [19, 20][2]. Therefore, the main aim of this paper is to extend the notion of τ-function to the quantum case. To this end, we are going to start with a detailed discussion of the usual τ-function and, in particular, to point out related group theory (symmetry) structures. Following this way, we are naturally led to the notion of the *generalized* τ-function that can be still algebraically described and satisfies a set of Hirota-like bilinear identities.

One can hardly over-estimate the necessity of the generalization of the τ-function notion. To begin with, the matrix models corresponding to the standard hierarchies are just the simplest models. More involved and physically important examples include the generalized Kontsevich-Kazakov-Migdal model [22], higher dimensional gauge theories [23], $2d$ gravity coupled to the conformal matter with $c > 1$ (i.e. with space-time dimension $d > 2$ which again corresponds to the higher-dimensional models) etc. In all these examples, the angular variables of the matrices can not be factorized out and then integrated out. Thus, they are described by more complicated "generalized" hierarchies.

Another important class of the generalized integrable systems has to correspond to the WZWN theories with the level greater that 1. Further moving into this direction should lead to the 2-loop, 3-loop etc. algebras within the integrable framework. All the listed examples are associated with the so called non-Cartanian hierarchies [13].

Yet another important problem to be solved in the course of quantizing the generalized τ-functions is the careful regularization of the Liouville theory [24]. This problem can be also differently approached, as we discuss later.

Let us remark here that we do not discuss why the q-deformation described in the paper has something to do with quantizing the system, i.e. with building quantum field theory (quantum mechanics) in place of classical one. Instead, we address to the papers [19, 20] that discuss why turning to quantum groups and quantum R-matrices corresponds to quantizing classical systems.

It was mentioned above that the main tool in studies of the generalized τ-functions is group theory. More concretely, one can use different symmetry structures of the integrable hierarchies

[2]See, however, very interesting papers [21].

to formulate them in algebraic terms. It is typical for the theoretical physics that the symmetry approach allows one to derive the general laws that do not depend on dynamics. In particular, it is the general symmetry structure that should explain the integrability of effective actions in quantum field theories. This is why one needs to reveal different algebraic structures and their role for the quantization.

As a typical example, in this paper that is a review of the papers [25]-[31] we consider either the Toda integrable system, or its particular case – the Liouville system. In sections 3 and 6, it is demonstrated that there are three different group structures in these systems. Two of them are related with the group action in the spectral parameter space: one is the group $GL(\infty)$ whose Grassmannian parametrizes all solutions of the $2d$ Toda hierarchy of the general form, the other one is the group $SL(n)$ that describes the reduction of this whole large space of solutions to the solutions of the $SL(n)$ Toda. Thus, both these groups act on solutions of the integrable hierarchy. It is these groups that are substituted by their quantum counterparts in the course of quantization.

Let us note, however, that the quantization procedure is not always a drill, that is to say, not any group structure would be automatically substituted by its quantum counterpart. Indeed, in the Liouville type system there is yet another group acting in "the space-time", i.e. on times of integrable system. In the course of quantization, this group still remains classical although is differently treated – the orbit interpretation is involved. In fact, if the classical system is obtained by the Hamiltonian reduction of a free system given on the cotangent bundle to a simple real Lie group [32], the quantum model is related to an irreducible unitary representation of *the same* group. Therefore, the quantum system is rather treated within the geometrical quantization framework [33]. This group structure is discussed in section 6. In the section, it is also demonstrated that the group theory methods allows one not only to construct the wave function of the quantum Toda system (that is called Whittaker function) [34], but also to calculate its asymptotics [35] and, therefore, S-matrix (i.e. two-point correlation function).

Such calculations are known for a while [36] but they have been performed using the Iwasawa decomposition that is inconvenient for the extension to the affine case. In section 6, we present alternative calculation that is based on the Gauss decomposition, which is naturally generalizable to the Liouville field theory described by the affine algebra (in the affine case, one needs to use the "point-wise" Gauss decomposition analogous to that used when bosonizing the WZWN theories [37]). All the calculations of the wave function and its asymptotics are immediately continued to this (affine) case giving the S-matrix of the $2d$ Liouville theory. This S-matrix coincides with that obtained recently by A. and Al.Zamolodchikovs [38] and H.Dorn and H.-J.Otto [39].

The other group structures – those q-deformed in the course of quantization – are discussed in the other sections. In particular, in section 4, the generalized τ-functions admitting immediate quantization are discussed. It is natural to require for such τ-functions still to remain c-numbers. In s.4.5 such a deformation of the classical differential hierarchy is, indeed, constructed so that the resulting hierarchy becomes a difference one. Nevertheless, this hierarchy turns out to be too trivial, since it can be obtained by a (quite complicated) redefinition of times of the original hierarchy. Therefore, the problem of constructing a non-trivial deformation of the τ-function (that leads to a non-commutative object) still persists.

This problem is solved, however, in the pure algebraic terms – in the same section 4, given a representation of any algebra, the τ-function is defined as the generating function of its matrix elements in some (arbitrary) representation. Thus defined τ-function satisfies a system of bilinear identities (BI). In the classical case, these BI are differential ones. To quantize the system, it suffices to substitute the classical group by its quantum counterpart. This makes the differential BI difference and the τ-function itself – operator.

Note that the τ-function defined by an arbitrary representation of algebra leads to BI given by *non-commutative* flows, that is to say, the theory is no longer integrable in the usual (Liouville)

sense. However, it still preserves many important features of the integrable system and corresponds to the non-Cartanian hierarchies [13] that we mentioned above.

In order to obtain the τ-function corresponding to the standard KP/Toda hierarchy, one needs to consider the fundamental representations of the group $SL(\infty)$. For these representations, there exists a special reduction of the general τ-function, which leads to the standard integrable hierarchies.

The next problem is to construct some quantum counterpart of the KP hierarchy. This problem is highly non-trivial, since the mentioned special reduction of the general τ-function admissible in the fundamental representations does not endure the quantization. Nevertheless, one can construct the q-deformation of some new classical hierarchy that is, by essence, the KP hierarchy but with unusual evolution. This is discussed in section 5.

For the sake of continuity of the paper, we place more technical comments in Appendices. Some additional details that are not included into this review can be found in the original papers [25]-[31] and, especially, [30], since the results of this latter are considered here very briefly.

The author is grateful to A.Gerasimov, S.Kharchev, S.Khoroshkin, D.Lebedev, A.Marshakov, A.Morozov, M.Olshanetsky and L.Vinet for the collaboration and useful discussions and to G.Weight, A.Gorsky, V.Dobrev, A.Zabrodin, Al.Zamolodchikov, V.Korepin, N.Nekrasov, N.Slavnov, I.Tyutin and J.Schnittger for the useful discussions. The work is partially supported by grants RFBR-96-01-01106 and INTAS-97-1038.

2 Fermionic representation for integrable hierarchies

We start with a brief description of the fermionic approach to the standard integrable hierarchies that is mostly due to the Kyoto school [40]-[44] (see also [45, 46]) and describe in these terms the important particular case of the semi-infinite (forced) hierarchies[3]. Note that the fermionic approach is the most universal language of those applied so far to describe the integrable systems, since it corresponds to the formal (Sato [47]) Grassmannian given in terms of the formal series. Although alternative approaches (see, e.g., [48]) suit better, say, the finite-gap solutions [49], it is the Sato Grassmannian that is necessary for dealing with the singular points (describing, for instance, the matrix models, see [50]).

2.1 General fermionic description of the standard integrable hierarchies

We begin with the description of the two-dimensional lattice Toda (2TDL) hierarchy. Its τ-function is defined through correlators in the theory of free fermions $\psi, \widetilde{\psi}$ (b, c-system of spin 1/2). Namely, the τ-function is the ratio of the correlators[4]

$$\tau_n(t, \bar{t} \mid G) \equiv \frac{\langle n \mid e^{H} G e^{\overline{H}} \mid n \rangle}{\langle n \mid G \mid n \rangle} \tag{2.1}$$

in the theory of the free fermionic fields $\psi(z)$, $\psi^*(z)$:

$$\begin{aligned} \psi(z) &= \sum_{\mathbf{Z}} \psi_n z^n \\ \psi^*(z) &= \sum_{\mathbf{Z}} \psi_n^* z^{-n-1} \end{aligned} \tag{2.2}$$

[3]Right these hierarchies describe, among other, the matrix models.

[4]Later on, we omit from this definition the denominator correlator (which is nothing but a normalization factor) wherever it is non-singular and, therefore, inessential.

$$\{\psi_k, \psi_m^*\} = \delta_{km}, \quad \{\psi_k, \psi_m\} = \{\psi_k^*, \psi_m^*\} = 0 \tag{2.3}$$

In the definition (2.1) we use the following notations

$$H = \sum_{k>0} t_k J_k; \quad \overline{H} = \sum_{k>0} \overline{t}_k J_{-k} \tag{2.4}$$

where the currents are defined as

$$J(z) = \psi^*(z)\psi(z), \quad \text{i.e.} \quad J_k = \sum_{\mathbf{Z}} \psi_i \psi_{i+k}^* \tag{2.5}$$

The quantity G is given by the formula

$$G = :\exp\left\{\sum_{m,n} \mathcal{G}_{mn}\psi_m^*\psi_n\right\}: \tag{2.6}$$

and is an element of the group $GL(\infty)$ (or the Clifford group) realized in the infinite dimensional Grassmannian (see also equivalent descriptions in [48]). The normal ordering is to be understood here as ordering w.r.t. the vacuum $|0\rangle$ defined by the conditions

$$\psi_m|0\rangle = 0 \quad m < 0 \ , \ \psi_m^*|0\rangle = 0 \quad m \geq 0 \tag{2.7}$$

One can also consider a more general normal ordering that is defined w.r.t. the vacuum $|k\rangle$ given by the conditions

$$\psi_m|k\rangle = 0 \quad m < k \ , \ \psi_m^*|k\rangle = 0 \quad m \geq k \tag{2.8}$$

Any concrete solution to the hierarchy (τ-function (2.1)) depends only on the choice of element G (or, equivalently, can be uniquely defined by the matrix $\mathcal{G}_{km}$). The remarkable result by the Kyoto school is the statement that any solution to the 2TDL hierarchy is given by an element G of the form (2.6), and, inversely, any such element G defines some solution to the 2TDL hierarchy.

Note that, within the fermionic approach [40]-[44], 2TDL is the most general one-component hierarchy. Say, the KP hierarchy can be obtained from the 2TDL hierarchy by canceling all the negative times and the zero time. One can also arbitrarily fix these times and do not include the corresponding evolution equations. In this respect, the KP hierarchy is a sort of sub-hierarchy of 2TDL, not a reduction, and is described by the same set of elements G with less number of the evolution flows.

Another example is the Toda chain which is already a reduction that can be obtained from 2TDL by special restricting element of the Grassmannian. As a result, the τ-function of the Toda chain depends not on the positive and negative times separately but only on their sums [45][5] (therefore, one can consider only one set of the evolution flows). This property of the Toda chain can be taken as its definition.

Now we derive some useful properties of the described system. One can easily get from the commutation relations (2.3) the transformation law of the fermionic modes w.r.t. the action of the element of the Grassmannian (2.6):

$$G\psi_k G^{-1} = \psi_j R_{jk} \ , \ G\psi_k^* G^{-1} = \psi_j^* R_{kj}^{-1} \tag{2.9}$$

where the matrix R_{jk} can be expressed through $\mathcal{G}_{jk}$ (see [10]). It will be clear later that R_{jk} is an important building block for determinant representations of the τ-functions.

Let us introduce more notations. Using (2.3), one can define evolution of $\psi(z)$ and $\psi^*(z)$ w.r.t. the times flows $\{t_k\}$, $\{\overline{t}_k\}$:

$$\psi(z,t) \equiv e^{H(t)}\psi(z)e^{-H(t)} = e^{\xi(t,z)}\psi(z) \tag{2.10}$$

[5] In most papers, the Toda chain depends only on *difference* of times. It is due to the opposite sign of $\overline{H}$ used in (2.1).

$$\psi^*(z,t) \equiv e^{H(t)}\psi^*(z)e^{-H(t)} = e^{-\xi(t,z)}\psi^*(z) \tag{2.11}$$

$$\psi(z,\bar{t}) \equiv e^{\overline{H}(\bar{t})}\psi(z)e^{-\overline{H}(\bar{t})} = e^{\xi(\bar{t},z^{-1})}\psi(z) \tag{2.12}$$

$$\psi^*(z,\bar{t}) \equiv e^{\overline{H}(\bar{t})}\psi^*(z)e^{-\overline{H}(\bar{t})} = e^{-\xi(\bar{t},z^{-1})}\psi(z) \tag{2.13}$$

where

$$\xi(t,z) = \sum_{k=1}^{\infty} t_k z^k \tag{2.14}$$

Let us also define the Schur polynomials

$$\exp\left\{\sum_{k>0} t_k x^k\right\} \equiv \sum_{k>0} P_k(t_k)x^k, \qquad P_k(t_k) = 0, \quad k<0 \tag{2.15}$$

Using this definition and (2.10)-(2.13), one immediately gets the evolution of the fermionic modes:

$$\psi_k(t) \equiv e^{H(t)}\psi_k e^{-H(t)} = \sum_{m=0}^{\infty} \psi_{k-m}P_m(t) \tag{2.16}$$

$$\psi^*_k(t) \equiv e^{H(t)}\psi^*_k e^{-H(t)} = \sum_{m=0}^{\infty} \psi^*_{k+m}P_m(-t) \tag{2.17}$$

$$\psi_k(\bar{t}) \equiv e^{\overline{H}(\bar{t})}\psi_k e^{-\overline{H}(\bar{t})} = \sum_{m=0}^{\infty} \psi_{k+m}P_m(\bar{t}) \tag{2.18}$$

$$\psi^*_k(\bar{t}) \equiv e^{\overline{H}(\bar{t})}\psi^*_k e^{-\overline{H}(\bar{t})} = \sum_{m=0}^{\infty} \psi^*_{k-m}P_m(-\bar{t}) \tag{2.19}$$

Now we can derive a determinant representation for the τ-function[6]. To this end, define the completely filled vacuum state $|-\infty\rangle$ that satisfies the requirement

$$\psi^*_i|-\infty\rangle = 0 \ , \ i \in \mathbf{Z} \tag{2.20}$$

Then, any "shifted" vacuum is obtained from the completely filled one as follows:

$$|n\rangle = \psi_{n-1}\psi_{n-2}...|-\infty\rangle \tag{2.21}$$

Note that any element G of the Clifford group (and, therefore, $e^{\overline{H}(\bar{t})}$) acts on $|-\infty\rangle$ trivially: $G|-\infty\rangle \sim |-\infty\rangle$. Therefore, using (2.17) and (2.18), one easily gets from (2.1):

$$\begin{aligned}\tau_n(t,\bar{t}) = \langle -\infty|...\psi^*_{n-2}(-t)\psi^*_{n-1}(-t)G\psi_{n-1}(\bar{t})\psi_{n-2}(\bar{t})...|-\infty\rangle \sim \\ \sim \det[\langle -\infty|\psi^*_i(-t)G\psi_j(\bar{t})G^{-1}|-\infty\rangle]\,|_{i,j\le n-1}\end{aligned} \tag{2.22}$$

Using (2.9), we now see that

$$G\psi_j(\bar{t})G^{-1} = \sum_{m,k} P_m(\bar{t})\psi_k R_{k,j+m} \tag{2.23}$$

and, therefore, the "manifest" solution to the 2TDL hierarchy reads in the determinant form as

$$\tau_n(t,\bar{t}) \sim \det\nolimits_{i,j<0} C_{i+n,j+n}(t,\bar{t}) \tag{2.24}$$

[6]The discussion below requires, certainly, in real calculations some convergency properties, in particular, of the determinants of infinite-dimensional matrices. It suffices to consider the matrices with finite number of non-zero diagonals [43]. However, sometimes this class of matrices turns out to be too restrictive like being the case in matrix models [51, 50]. On the other hand, the formulas below are direct implications of some relations that can be checked out for formal series and, therefore, are always correct once the convergency conditions are fulfilled. By the same reason, we ignore the issue of existence of the completely filled vacuum state below.

where[7]

$$C_{ij}(t,\bar{t}) = \sum_{k,m} R_{km} P_{k-i}(t) P_{m-j}(\bar{t}) \tag{2.25}$$

Let us note that the time dependence of the matrix elements (2.25) in the determinant (2.24) is given by the following equations:

$$\partial C_{ij}/\partial t_p = C_{i,j-p},\ j > p > 0 \tag{2.26}$$

$$\partial C_{ij}/\partial \bar{t}_p = C_{i-p,j},\ i > p > 0 \tag{2.27}$$

which immediately follows from the corresponding property of the Schur polynomials:

$$\partial P_k/\partial t_p = P_{k-p} \tag{2.28}$$

that is a direct consequence of their definition.

It has been already remarked that the KP hierarchy is given by the evolution only w.r.t. the positive times $\{t_k\}$, while the negative times $\{\bar{t}_k\}$ are just parameters which give a set of points in the Grassmannian and can be removed by a redefinition of the matrix R_{km}. Then, the τ-function of the (modified) KP hierarchy reads as

$$\tau_n(t) = \langle n|e^{H(t)}G(\bar{t})|n\rangle \sim \det_{i,j<0}\left[\sum_k R_{k,j+n}(\bar{t})P_{k-i-n}(t)\right] \tag{2.29}$$

where $G(\bar{t}) \equiv Ge^{\overline{H}(\bar{t})}$ and

$$R_{kj}(\bar{t}) \equiv \sum_m R_{km}P_{m-j}(\bar{t}) \tag{2.30}$$

2.2 Bilinear identities for the classical 2TDL hierarchy

For future use, we need two more *crucial* formulas in the theory of integrable systems [41]:

$$\langle n|\psi(z)\exp[H(t)] = z^{n-1}\langle n-1|\exp[H(t-\epsilon(z^{-1}))] \equiv z^{n-1}\widehat{X}(z,t)\ \langle n-1|e^{H(t)} \tag{2.31}$$

$$\langle n|\psi^*(z)\exp[H(t)] = z^{-n}\langle n+1|\exp[H(t+\epsilon(z^{-1}))] \equiv z^{-n}\widehat{X}^*(z,t)\langle n+1|e^{H(t)} \tag{2.32}$$

where

$$\widehat{X}(z,t) \equiv e^{\xi(z,t)}\ e^{-\xi(z,\widetilde{\partial}_t)} \tag{2.33}$$

$$\widehat{X}^*(z,t) \equiv e^{-\xi(z,t)}\ e^{\xi(z,\widetilde{\partial}_t)} \tag{2.34}$$

where as usual $\widetilde{\partial}_t \equiv (\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \ldots)$.

Using these formulas, one can express the Baker-Akhiezer (BA) function of the KP hierarchy, defined via τ-function by the formula [41]

$$\Psi(\mu|t_n) = e^{\Sigma t_n\mu^n}\frac{\tau_0(t_n - \frac{\mu^{-n}}{n}\Big|\,\bar{t}_n)}{\tau_0(t_n|\bar{t}_n)} \tag{2.35}$$

through the fermionic correlator:

$$\Psi(\mu|t_n) = \frac{\left\langle 1\left|e^{H(t)}\psi(\mu)G\right|0\right\rangle}{\langle 0\,|e^{H(t)}G|\,0\rangle} \tag{2.36}$$

[7]Note that throughout we ignore the summation limits, since they are automatically reproduced by the property (definition) of the Schur polynomials $P_k = 0$ at $k < 0$ in (2.15).

Analogously, one can consider the conjugated BA function given by

$$\Psi^*(\mu|t_n) = \frac{\left\langle 1 \left| e^{H(t)} \psi^*(\mu) G \right| 0 \right\rangle}{\langle 0 | e^{H(t)} G | 0 \rangle} \tag{2.37}$$

Involving "non-zeroth" vacuums, one can also introduce more general pair of the BA functions depending on the zero time n.

Considering the "whole" 2TDL hierarchy, one can reasonably introduce *four* different BA functions, since, in this case, it makes sense to put fermions both to the left and to the right of the Grassmannian element G.

Now let us derive the bilinear identities (BI) satisfied by the τ-function of the 2TDL hierarchy (2.1) [43, 45]. To this end, note that due to (2.9) the tensor product of the Grassmannian elements $G \otimes G$ (2.6) commutes with the tensor product $\Gamma \equiv \sum_i \psi_i \otimes \psi_i^*$. To make our notations consistent with the forthcoming sections, we denote hereafter the fermionic modes ψ_k through ψ_k^+ and ψ_k^* through ψ_k^-.

Look now at the matrix elements of the operator identity

$$\Gamma(G \otimes G) = (G \otimes G)\Gamma \tag{2.38}$$

taken in between the states $\langle n+1|U(t) \otimes \langle m-1|U(t')$ and $\overline{U}(\overline{t})|n\rangle \otimes \overline{U}(\overline{t}')|m\rangle$, where we denote $U(t) \equiv e^{H(t)}$ and $\overline{U}(\overline{t}) \equiv e^{\overline{H}(\overline{t})}$:

$$\begin{aligned} &\sum_i \langle n+1|U(t)\psi_i^+ G\overline{U}(\overline{t})|n\rangle \cdot \langle m-1|U(t')\psi_i^- G\overline{U}(\overline{t}')|m\rangle = \\ &= \sum_i \langle n+1|U(t)G\psi_i^+ \overline{U}(\overline{t})|n\rangle \cdot \langle m-1|U(t')G\psi_i^- \overline{U}(\overline{t}')|m\rangle \end{aligned} \tag{2.39}$$

Then, one can rewrite (2.39) through the free fermion fields (2.2):[8]

$$\begin{aligned} &\oint_\infty dz \langle n+1|U(t)\psi^+(z) G\overline{U}(\overline{t})|n\rangle \cdot \langle m-1|U(t')\psi^-(z) G\overline{U}(\overline{t}')|m\rangle = \\ &= \oint_0 dz \langle n+1|U(t)G\psi^+(z)\overline{U}(\overline{t})|n\rangle \cdot \langle m-1|U(t')G\psi^-(z)\overline{U}(\overline{t}')|m\rangle \end{aligned} \tag{2.40}$$

Generalizing definitions of the BA functions (2.36) and (2.37) to the non-zero vacuum and involving negative times

$$\Psi_n^{\pm,i} \equiv \langle n \pm 1|\widehat{U}(t)\psi_i^\pm G\widehat{\overline{U}}(\overline{t})|n\rangle \tag{2.41}$$

one can rewrite these formulas in a more compact form (hereafter we use the notation t both for positive and for negative times all together when it can not mislead):

$$\sum_i \Psi_k^{+,i}(t)\Psi_l^{-,i}(t') = \sum_j \overline{\Psi}_{k+1}^{+,j}(\overline{t})\overline{\Psi}_{l-1}^{+,j}(\overline{t}') \tag{2.42}$$

Here $\overline{\Psi}$ gives the second pair of the BA functions defined by the correlator with the fermion to the right of the element G.

These BA functions are generated by the action of the vertex operators (2.33)-(2.34) (see (2.31)-(2.32):

$$\sum_i \Psi_k^{+,i}(t)z^i \equiv \widehat{X}^+(z,t)\tau_n(t), \quad \sum_i \Psi_k^{-,i}(t)z^{-i-1} \equiv \widehat{X}^-(z,t)\tau_n(t) \tag{2.43}$$

[8]The integration contour in this formula depends on whether the integrand is expanded into positive or negative powers of z.

and analogously for $\widehat{\widehat{X}}^{\pm}(z,t)$. Then, (2.39) can be rewritten as

$$\oint_{\infty} dz \widehat{X}^{-}(z,t)\tau_n(t,\bar{t})\widehat{X}^{+}(z,t')\tau_m(t',\bar{t}') = \oint_0 dz \widehat{\widehat{X}}^{-}(z,\bar{t})\tau_{n+1}(t,\bar{t})\widehat{\widehat{X}}^{+}(z,\bar{t}')\tau_{m-1}(t',\bar{t}') \tag{2.44}$$

This integral BI can be reduced to the infinite set of equations (see also (3.4)) by expanding into degrees of $t-t'$ and $\bar{t}-\bar{t}'$:

$$\begin{gathered}\sum_{i=0}^{\infty} P_i(-2y)P_{i+1}(\widetilde{D}_t)e^{[\sum_i y_i D_{t_i}]}\tau\cdot\tau = 0\\ \tau_n\partial_{t_1}\partial_{\bar{t}}\tau_n - \partial_{t_1}\tau_n\partial_{\bar{t}_1}\tau_n = \tau_{n+1}\tau_{n-1}\end{gathered} \tag{2.45}$$

where D_t is the Hirota symbol given by its action onto the product of two functions: $D_x^k f\cdot g \equiv \partial_y[f(x+y)g(x-y)]\Big|_{y=0}$, $\widetilde{D}\equiv(D_{t_1},\ \frac{1}{2}D_{t_2},\ldots)$, P_i's are Schur polynomials. Expanding these equations (2.45) w.r.t. the arbitrary set of parameters $\{y_k\}$, one finally arrives at the KP/Toda hierarchy equations.

2.3 Fermionic representation for the forced hierarchies

Now we apply the described formalism to the case of the so-called forced hierarchies and obtain, in particular, determinant formulas of [52, 46]. The main difference of the formulas like (2.24) with the determinant representations to be obtained in this subsection is that the forced hierarchies corresponds to the determinants of the *finite* matrices. This important difference is due to the very specific boundary condition (from the point of view of the particle chain)

$$\tau_0 = 1 \tag{2.46}$$

This condition may serve as a definition of the forced hierarchies [53] investigated in [52, 46] in detail. It turns out that the forced hierarchies are described by singular elements of the Grassmannian, which depend on only positive (or only negative) fermionic modes. It implies "quarter-infinite" matrices $\mathcal{G}_{mn}$ in (2.6) and, indeed, leads to the determinants of finite matrices. We describe the forced hierarchies in this subsection very briefly leaving some technical details till Appendix 1. More details can be found in [52, 46].

Thus, we are looking for the point of the Grassmannian in the form

$$G = G_0 P_+ \tag{2.47}$$

where P_+ is the projector onto the positive states:

$$P_+|n\rangle = \theta(n)|n\rangle \tag{2.48}$$

This projector admits a natural fermionic realization

$$P_+ =: \exp\left[\sum_{i<0}\psi_i\psi_i^*\right]: \tag{2.49}$$

and possesses a set of properties

$$P_+\psi_{-k}^* = \psi_{-k}P_+ = 0\ ,\ k>0 \tag{2.50}$$

$$[P_+,\psi_k] = [P_+,\psi_k^*] = 0\ ,\ k\geq 0 \tag{2.51}$$

$$P_+^2 = P_+ \tag{2.52}$$

If the correlator in (2.1) contains this projector, we naturally require for G_0 to depend only on ψ_k and ψ_k^* with $k \geq 0$. We fix its form to be

$$G_0 =: \exp \left\{ \left(\int_\gamma A(z,w)\psi_+(z)\psi_+^*(w^{-1})dzdw \right) - \sum_{i\geq 0} \psi_i\psi_i^* \right\} : \tag{2.53}$$

where γ is an integration domain. The key formula (we leave its proof till Appendix 1) gives the τ-function corresponding to the element of the Grassmannian choosen in the form (2.47):

$$\begin{gathered} \tau_n(t,\bar{t}) = \langle n|e^{H(t)}G_0P_+e^{\overline{H}(\bar{t})}|n\rangle = \\ = \frac{1}{n!}\int_\gamma \Delta(w)\Delta(z)\prod_{i=1}^n A(z_i,w_i)e^{\xi(t,z_i)+\xi(\bar{t},w_i)}dz_idw_i \end{gathered} \tag{2.54}$$

where $\Delta(z) \equiv \det_{i,j} z_j^{i-1}$ is the Van-der-Monde determinant.

Repeating the above calculation (see also Appendix 1), one can get the determinant representation for the forced hierarchy:

$$\tau_n(t,\bar{t}) = \det \left[\partial_{t_1}^i \partial_{\bar{t}_1}^j \int_\gamma A(z,w)e^{\xi(t,z)+\xi(\bar{t},w)}dzdw \right] \Bigg|_{i,j=0,\ldots,n-1} \tag{2.55}$$

This is, in fact, the determinant of the finite matrix, which provides, in particular cases, the determinant representations for the matrix models [46].

Note that the matrix A_{ij} defined by the expansion of $A(z,w)$ into degrees of z and w coincides with the matrix R_{ij} from (2.9) (see Appendix 1). This simple relation of the fermion rotation matrix and the matrix giving the element of the Grassmannian is specific for the forced hierarchies.

The element of the Grassmannian determined in (2.47) is in no way unique, since there are many possibilities to continue the relation (2.46) to the negative values of n. The choice we used corresponds to the condition

$$\tau_n = 0 \quad \text{at} \quad n < 0 \tag{2.56}$$

This way of defining the τ-function is advantageous when reducing to the Toda chain, since, in this case, the τ-function depends only on the sum of times $t_k + \bar{t}_k$ that is the defining property of the corresponding infinite hierarchy. Other important choices for the element of the Grassmannian G have been discussed in [52]. If considering only dependence on times t_k, one of interesting choices corresponds to the condition

$$\tau_n(t_k) = \tau_{-n}\left((-)^k t_k\right) \tag{2.57}$$

and describes the (modified) CKP hierarchy [42], i.e. the corresponding element of the Grassmannian belongs to $Sp(\infty)$. Another choice of this element is just $G = G_0$ that corresponds to the condition

$$\tau_n = 1 \quad \text{at} \quad n < 0 \tag{2.58}$$

etc.

3 Group theory structures of the Toda molecule

Before generalizing the construction of the previous section, we discuss group theory structures that are presented in integrable systems and take as our main example the Liouville (Toda) theory[9]. Note that it often happens for *different* group structures to be presented in the *same*

[9]In fact, it has been realized already quite a while ago [54, 55] (see also, by the appearance, the first paper on this stuff [56]) that the group theory approach is one of the most effective and elegant frameworks for classical integrable systems.

integrable system. In particular, some groups can act in the solution space of integrable hierarchy, while other ones – just on the variables of integrable equations (in "the space-time") [31]. We start with considering the structures of the first type, leaving those of the second type till section 4.

As it was discussed in the previous section, all solutions of the integrable 2DTL hierarchy (2.1) are described by the group elements of $GL(\infty)$ (2.6). This group acts in the space of solutions and simultaneously in the space of the spectral parameter. This is the group structure we are mainly interested in, since just this group structure is in charge of the majority of defining properties of integrable system and it is deformed in the course of quantizing. However, in the example we are going to look at now, that is the Liouville system, there exists yet another group.

We begin with a natural restriction of the forced hierarchies given by the condition (2.56) onto finite systems. To this end, we impose one additional constraint

$$\tau_n = 0, \quad n > p \tag{3.1}$$

for some p. This system corresponds to the two-dimensional $(p-1)$-body Toda molecule [57]-[58] and is sometimes called non-periodic $SL(p)$ Toda chain [54]. Since it is just a very particular case of the forced hierarchy, one can make use of the formulas of the previous section and Appendix 1. Then it is immediate to check that the $SL(p)$ Toda molecule is described by the arbitrary fermion rotation matrix R_{ij} only restricted to be of rank p [27]. This means that the matrix can be presented in the form

$$R_{ij} = \sum_k^p f_i^{(k)} g_j^{(k)} \tag{3.2}$$

$f_i^{(k)}$ and $g_j^{(k)}$ being arbitrary coefficients and the kernel of the Grassmannian element $A(z,w)$ (A1.8) (whose power series expansion into powers z and w gives the matrix R_{ij}) being of the form

$$A(z,w) = \sum_k^p f^{(k)}(z) g^{(k)}(w) \tag{3.3}$$

where $f^{(k)}(z)$ and $g^{(k)}(z)$ are arbitrary functions.

The simplest way to check this statement is to use the determinant representation (A1.25). In fact, from the first equation of the 2DTL hierarchy

$$\tau_n \partial_{t_1} \partial_{\bar{t}_1} \tau_n - \partial_{t_1} \tau_n \partial_{\bar{t}_1} \tau_n = \tau_{n+1} \tau_{n-1} \tag{3.4}$$

and condition (3.1), one can establish that $\log\tau_0$ and $\log\tau_p$ satisfy the free wave equation

$$\partial_{t_1} \partial_{\bar{t}_1} \log\tau_0 = \partial_{t_1} \partial_{\bar{t}_1} \log\tau_p = 0 \tag{3.5}$$

Since the relative normalization of τ_n is not fixed yet, one can always choose $\tau_0 = 1$. Then,

$$\tau_0(t) = 1 \ , \qquad \tau_p(t) = \chi(t_1)\overline{\chi}(\bar{t}_1) \tag{3.6}$$

where $\chi(t_1)$ and $\overline{\chi}(\bar{t}_1)$ are arbitrary functions. The two-dimensional Toda system with the boundary conditions (3.6) has been considered in [57]. Solution to the equation (3.4) is, in this case, [58]:

$$\tau_n(t) \ = \ \det \partial_{t_1}^{i-1} (-\partial_{\bar{t}_1})^{j-1} \tau_1(t) \tag{3.7}$$

with

$$\tau_1(t) = \sum_{k=1}^p a^{(k)}(t) \overline{a}^{(k)}(\bar{t}_1) \tag{3.8}$$

where functions $a^{(k)}(t)$ and $\overline{a}^{(k)}(\overline{t}_1)$ satisfy the conditions

$$\det\partial_{t_1}^{i-1}a^{(k)}(t)=\chi(t)\ ,\ \ \det(-\partial_{\overline{t}_1})^{i-1}\overline{a}^{(k)}(\overline{t}_1)=\overline{\chi}(\overline{t}_1) \tag{3.9}$$

This result corresponds to formula (A1.25) with the kernel $A(z,w)$ of the form (3.3). Let us draw that, although the Toda molecule looks like a forced hierarchy with one additional projector, it is generically described by the infinite number of the fermion modes because of infinitely many non-vanishing elements of the matrix (3.2).

Note that the function τ_p can be put equal to unity. This is exactly the choice corresponding to the group $SL(p)$, while the general solution is associated with the group $GL(p)$.

Let us discuss now how one can observe this group structure in the Toda molecule. For the sake of simplicity, we consider below the instance of the Liouville system ($p=2$), the extension to the higher groups being quite immediate. The simplest way of doing is to restore the manifest group structure in (3.7)-(3.8). This has been done in [59]. Moreover, this representation admits the deformation so that the quantum Liouville system is merely described by the quantum group $SL_q(2)$ [60, 61]. Still, it is more convenient for us to employ here a little bit different approach to the problem [27].

Namely, we are going to look at the Liouville theory as at a reduction of the general two-dimensional Toda system associated with the group $GL(\infty)$. In this case, the functions $f^{(k)}(z)$ and $g^{(k)}(z)$ (either the set of the corresponding coefficients $f_i^{(k)}$ and $g_i^{(k)}$) describe the way for the group $SL(2)$ associated with the Liouville system to embed into the group $GL(\infty)$ in the course of reduction. Thus, the different embeddings are related by external $GL(\infty)$-automorphisms of the group $SL(2)$.

Let us describe it in more explicit terms. The Liouville equation is of the form:

$$\partial\tau_1\overline{\partial}\tau_1-\tau_1\partial\overline{\partial}\tau_1=\tau_0\tau_2=1 \tag{3.10}$$

or

$$\partial\overline{\partial}\phi=2e^{\phi},\quad \tau_1=e^{-\phi/2} \tag{3.11}$$

Its general solution

$$\tau_1(t,\overline{t}|g)=(1+A(t)B(\overline{t}))\left[\frac{\partial A}{\partial t}\frac{\partial B}{\partial\overline{t}}\right]^{-\frac{1}{2}} \tag{3.12}$$

is parametrized by two arbitrary functions $A(t)$ and $B(\overline{t})$. These functions are related with the fermion rotation matrix via the following formulas that can be obtained by comparing (A1.25) and (3.12):

$$\begin{gathered}G\psi_iG^{-1}=\left(\int\int dt\frac{e^{-tx}}{\sqrt{\partial A}}dxx^{i-1}\right)\cdot\sum_k\left(\int\int d\overline{t}\frac{e^{-\overline{t}y}}{\sqrt{\overline{\partial}B}}dyy^{k-1}\right)\psi_k+\\+\left(\int\int dt\frac{e^{-tx}A}{\sqrt{\partial A}}dxx^{i-1}\right)\cdot\sum_k\left(\int\int d\overline{t}\frac{e^{-\overline{t}y}B}{\sqrt{\overline{\partial}B}}dyy^{k-1}\right)\psi_k\equiv f_i\Psi^{(1)}+g_i\Psi^{(2)}\end{gathered} \tag{3.13}$$

Thus, the element G of the Grassmannian, which is associated with the Liouville system rotates the fermionic modes in the two-dimensional space given by the "dressed" fermions $\Psi_i^{(1,2)}$ that evidently depend on the concrete choice of G, i.e. on the functions A and B. Each choice of G fixes such a pair of the fermions so that they give the embedding of the group $SL(2)$ into the group $GL(\infty)$ and all the variety of solutions to the Liouville equation is given by all possible choices of the functions A and B, i.e. by all possible embeddings of this type.

4 Generalized τ-function and bilinear identities

Our previous discussion of τ-functions was mainly concentrated on their determinant representations. However, the τ-function can be also introduced by the hierarchy of defining equations, namely, by an (infinite) set of bilinear identities of the Hirota type (3.4) and (2.45). We show in this section how the τ-function can be generalized still to satisfy an hierarchy of bilinear identities, and in the next section we discuss its determinant representations that turn out to be more particular property.

The main idea of our approach is to formulate any integrable hierarchy in pure algebraic terms so that it could be constructed for any (highest weight) representation of any group. In particular, quantizing means substituting the group by its quantum counterpart.

4.1 τ-function and bilinear identities

τ-function. Thus, for a given universal enveloping algebra (UEA) $U(\mathcal{G})$ (over some ring) and a Verma module V of this algebra, we define the τ-function as a generating function of all the matrix elements $\langle m|g|\overline{m}\rangle_V$ [26]:

$$\tau_V(t,\overline{t}|g) \equiv \sum_{m,\overline{m}\in V} s^V_{m,\overline{m}}(t,\overline{t})\ \langle m|g|\overline{m}\rangle_V \tag{4.1}$$

The main ambiguity of this definition is the choice of the function $s^V_{m,\overline{m}}(t,\overline{t})$, which is to be done in a "clever" way to have τ-function with good properties. The ambiguity is partially eliminated for the highest weight representations, when one can require

$$\tau_V(t,\overline{t}|g) =< 0_V|U(t)g\overline{U}(\overline{t})|0_V > \tag{4.2}$$

with some operators U and $\overline{U}$ that does not depend on V, $|0_V>$ being the highest weight vector[10].

This requirement naturally arises if one considers the τ-function of the 2DTL hierarchy that corresponds to $\mathbf{G} = SL(p)$ with V being one of the $p-1$ fundamental representations. We will return to this case in section 5.

However, the set of bilinear identities that we are going to derive now does not depend on the concrete choice of the functions $s^V_{m,\overline{m}}(t,\overline{t})$, thus, we will return to their explicit form later.

Intertwining operators and bilinear identities. Let us generalize now the derivation of the BI of section 2 to the case of the general τ-function (4.1). Its part that can be formulated in purely algebraic terms is the derivation of the counterpart of (2.39). The procedure of derivation consists now of some steps.

1. The starting point is to embed a Verma module $\widehat{V}$ into the tensor product $V\otimes W$, where W is some (arbitrary) finite-dimensional representation of $\mathcal{G}$.[11] With the fixed choice of V and W, there exist only finite number of possible $\widehat{V}$.

[10]Evolution operators $U(t)$ ($\overline{U}(\overline{t})$) for the general representation depend on all raising (lowering) generators of algebra, which generally do not commute. Therefore, the time flows in the hierarchy equations described below should not commute in contrast to the case of the standard integrable hierarchies. Still these evolution operators can commute in the specially choosen representations, for instance, in the fundamental representations of the classical groups – see section 6 for more details.

[11]In the affine case, one should use finite-dimensional (evaluation) representations as well. These are representations with zero central charge, and they are not highest weight representations – cf. the definition of vertex operators in [62, 63].

Define now some generalization of the fermions – intertwining operators. The right intertwining operator of the type W is defined to be a homomorphism of the $\mathcal{G}$-modules:

$$E_R: \quad \widehat{V} \longrightarrow V \otimes W \tag{4.3}$$

This intertwining operator can be constructed for the highest weight vectors

$$|\mathbf{0}\rangle_{\widehat{V}} = \left(\sum_{\{p_\alpha, i_\alpha\}} A\{p_\alpha, i_\alpha\} \left(\prod_{\alpha>0} (T_{-\alpha})^{p_\alpha} \otimes (T_{-\alpha})^{i_\alpha} \right) \right) |\mathbf{0}\rangle_V \otimes |\mathbf{0}\rangle_W \tag{4.4}$$

and then continued explicitly to the whole representation in accordance with formula:

$$\widehat{V} = \left\{ |\mathbf{n}_\alpha\rangle_{\widehat{V}} = \prod_{\alpha>0} (\Delta(T_{-\alpha})^{n_\alpha} |\mathbf{0}\rangle_{\widehat{V}} \right\} \tag{4.5}$$

where $T_{-\alpha}$ are (lowering) generators of algebra, which belong to its maximal negative (right) nilpotent subalgebra $\overline{N}(\mathcal{G})$, $\boldsymbol{\alpha}$ are the positive roots, the vacuum state is annihilated by all the raising operators T_α from the positive (left) nilpotent subalgebra $N(\mathcal{G})$, the Verma module is built by the action of all generators $T_{-\alpha}$ on the highest weight state: $V = \{|\mathbf{n}_\alpha\rangle_V = \prod_{\alpha>0} (T_{-\alpha})^{n_\alpha} |\mathbf{0}\rangle_V\}$. Formula (4.5) reflects the fact that the action of $\mathcal{G}$ on the tensor product of representations (Verma modules) is defined by the co-multiplication Δ and, for finite W, allows one to present every $|\mathbf{n}_\alpha\rangle_{\widehat{V}}$ as a *finite* sum of states $|\mathbf{m}_\alpha\rangle_V$ with the coefficients taking values in W.

2. As the next step, we consider another triple of modules that define the left intertwining operator

$$E'_L: \quad \widehat{V}' \longrightarrow W' \otimes V' \tag{4.6}$$

so that the product $W \otimes W'$ contains the *unit* representation of $\mathcal{G}$.

BI in terms of UEA. We turn now to the immediate derivation of the BI. They can be obtained in two forms – in the operator form that is analogous to (2.38) and as a relation for the matrix elements (i.e. in terms of the algebra of functions on the group). The first form can be obtained in two steps. As the first step, one needs to consider the projection of the product $W \otimes W'$ onto the unit representation

$$\pi: \quad W \otimes W' \longrightarrow I \tag{4.7}$$

that can be manifestly provided by multiplying any element from $W \otimes W'$ and

$$\pi = {}_W\langle\mathbf{0}| \otimes {}_{W'}\langle\mathbf{0}| \left(\sum_{\{i_\alpha, i'_\alpha\}} A_\pi\{i_\alpha, i'_\alpha\} \left(\prod_{\alpha>0} (T_{+\alpha})^{i_\alpha} \otimes (T_{+\alpha})^{i'_\alpha} \right) \right) \tag{4.8}$$

Using this projection, one can construct the new intertwining operator

$$\Gamma: \quad \widehat{V} \otimes \widehat{V}' \stackrel{E_R \otimes E'_L}{\longrightarrow} V \otimes W \otimes W' \otimes V' \stackrel{I \otimes \pi \otimes I}{\longrightarrow} V \otimes V' \tag{4.9}$$

possessing the property (2.38)

$$\Gamma(g \otimes g) = (g \otimes g)\Gamma \tag{4.10}$$

for any group element g such that

$$\Delta(g) = g \otimes g \tag{4.11}$$

Put it differently, the space $W \otimes W'$ contains the canonical element of pairing $w_i \otimes w^i$ commuting with the action of $\Delta(g)$. This means that the operator $\sum_i E_i \otimes E^i : V \otimes V' \longrightarrow \widehat{V} \otimes \widehat{V}'$ ($E_i \equiv E(w_i)$, $E^i \equiv E(w^i)$) commutes with $\Delta(g)$.

The identity (4.10) is the algebraic formulation of the BI. In order to obtain differential (or difference) identities, one needs (like it was in the above example of the 2DTL hierarchy) to use the definition (4.2) and consider the matrix element of the identity (4.10) along with the evolution operators between the highest weight states. This gives rise to identities (there is a lot of equivalent BI in accordance with a lot of possible choices of the representations V and $\widehat{V}$) for the objects like FBA, i.e. for the averages (4.2) with additional insertions of the intertwining operators E_i and E^i. Then, using the commutation relations of the intertwining operators with algebra generators, one can push E_i to the highest weight vector. The result of this procedure can be imitated by the action of some differential or difference operators that leads to the differential or difference BI. This latter step is, however, not always possible and requires the correct choice of the evolution operators $U(t)$, $\overline{U}(\overline{t})$. Up to now, there is no general recipe for such a choice. However, in the next two subsections we construct some explicit examples that illustrate how the choice is usually made. Certainly, the concrete differential (difference) form of the BI depends on the concrete algebra and its concrete representation and can not be obtained in a general algebraic form. Moreover, we will show soon that the same system can be described by different differential hierarchies – or even by difference and differential hierarchies – depending on the choice of the evolution operators.

BI in terms of the algebra of functions. To conclude the subsection, we obtain the BI in terms of matrix elements, i.e. in terms of the algebra of functions on the group. This language is dual to that of UEA.

Thus, let us write down the matrix element (4.10) taken in between four states

$$ {}_{V'}\langle k'|\,{}_V\langle k|(g\otimes g)\Gamma|n\rangle_{\widehat{V}}|n'\rangle_{\widehat{V}'} \;=\; {}_{V'}\langle k'|\,{}_V\langle k|\Gamma(g\otimes g)|n\rangle_{\widehat{V}}|n'\rangle_{\widehat{V}'} \tag{4.12} $$

Action of the operator Γ can be presented by the formula

$$ \Gamma|n\rangle_{\widehat{\lambda}}|n'\rangle_{\widehat{\lambda}'} = \sum_{l,l'} |l\rangle_\lambda |l'\rangle_{\lambda'}\Gamma(l,l'|n,n') \tag{4.13} $$

i.e. (4.12) goes into

$$ \sum_{m,m'} \Gamma(k,k'|m,m')\frac{||k||^2_\lambda ||k'||^2_{\lambda'}}{||m||^2_{\widehat{\lambda}}||m'||^2_{\widehat{\lambda}'}}\langle m|g|n\rangle_{\widehat{\lambda}}\langle m'|g|n'\rangle_{\widehat{\lambda}'} = \sum_{l,l'}\langle k|g|l\rangle_\lambda\langle k'|g|l'\rangle_{\lambda'}\Gamma(l,l'|n,n') \tag{4.14} $$

To rewrite this expression as some differential or difference equation, one needs to use formula (4.1) for the τ-function. Then, one can get the generating equation for the identities (4.14) making use of the explicit formulas for the matrix elements $\Gamma(l,l'|n,n')$ that can be calculated in the group theory framework (they are nothing but the corresponding Clebsh-Gordon coefficients). Again to present the generating equation in the differential form, one needs to choose correctly the coefficients $s^R_{m,\overline{m}}(t,\overline{t})$ in formula (4.1). Technically, the simplest way to make this choice is often to use concrete representation, i.e. BI formulated in terms of the algebra of functions. We illustrate this method in the next section.

To conclude this subsection, note that, in order to reproduce literally the derivation of the BI for the 2DTL hierarchy, one would introduce for *the same* triple of representations V, V', W the conjugated pair of the intertwining operators

$$ \Phi:\ \widehat{V}\otimes W\to V,\qquad \Phi^\star:\ V\to W\otimes\widehat{V} \tag{4.15} $$

that, by definition, satisfy the conditions

$$ \Delta(g)\Phi = \Phi g,\quad \Phi^\star\Delta(g) = g\Phi^\star \tag{4.16} $$

since being homomorphisms, and the algebra action in the tensor product is given by the co-multiplication. These conditions are the literal generalization of formula (2.9), and the canonical element commutes again with the group element g giving rise to the equation (4.10). Still, the proposed derivation of the BI using the pair of representation triples looks more general, and, hence, we use it throughout this section.

4.2 Example – $SL_q(2)$

To illustrate the above described rather abstract construction of the generalized τ-function and BI, consider now as an example the case of the quantum group $SL_q(2)$ [26]. This example, at the same time, illustrates what are the τ-function and BI in quantum integrable systems. It was already noted that, in the framework under consideration, quantizing just implies substituting group by its quantum counterpart. Note that, from now on, by q-deformation we assume the real $q > 1$, although other values of q can be also easily treated by the methods developed.

Bilinear identities. Algebra $U_q(SL(2))$ [12] is given by the generators T_+, T_- and T_0 subject to the commutation relations

$$q^{T_0}T_\pm q^{-T_0} = q^{\pm 1}T_\pm, \quad [T_+, T_-] = \frac{q^{2T_0} - q^{-2T_0}}{q - q^{-1}} \tag{4.17}$$

and by the co-multiplication

$$\Delta(T_\pm) = q^{T_0} \otimes T_\pm + T_\pm \otimes q^{-T_0}, \quad \Delta(q^{T_0}) = q^{T_0} \otimes q^{T_0} \tag{4.18}$$

The Verma module V_λ of the highest weight λ (not obligatory half-integer) consists of the elements

$$|n\rangle_\lambda \equiv T_-^n|0\rangle_\lambda, \quad n \geq 0 \tag{4.19}$$

such that

$$\begin{gathered} T_-|n\rangle_\lambda = |n+1\rangle_\lambda, \quad T_0|n\rangle_\lambda = (\lambda - n)|n\rangle_\lambda, \quad T_+|n\rangle_\lambda \equiv b_n(\lambda)|n-1\rangle_\lambda \\ b_n(\lambda) = [n]_q[2\lambda + 1 - n]_q, \quad [x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}}, \quad [n]_q! \equiv [1]_q[2]_q \dots [n]_q \\ ||n||_\lambda^2 \equiv {}_\lambda\langle n|n\rangle_\lambda = \frac{[n]_q!\ \Gamma_q(2\lambda+1)}{\Gamma_q(2\lambda+1-n)} \overset{\lambda \in \mathbf{Z}/2}{=} \frac{[2\lambda]_q![n]_q!}{[2\lambda - n]_q!} \end{gathered} \tag{4.20}$$

Now we obtain BI working in terms of the algebra of functions on the quantum group, i.e. manifestly calculating matrix elements of the operator Γ.

As the module W we choose the spin $\frac{1}{2}$ irrep of $U_q(SL(2))$. Then, $\widehat{V} = V_{\lambda\pm\frac{1}{2}}$, $V = V_\lambda$.

To calculate matrix elements of Γ, one needs to project the tensor product of two different W onto the singlet state $S = |+\rangle|-\rangle - q|-\rangle|+\rangle$:

$$(A|+\rangle + B|-\rangle) \otimes (|+\rangle C + |-\rangle D) \longrightarrow AD - qBC \tag{4.21}$$

With our choice of W, one needs to consider two different cases:

(A) $\widehat{V} = V_{\lambda-\frac{1}{2}}$ and $\widehat{V}' = V_{\lambda'-\frac{1}{2}}$, or

(B) $\widehat{V} = V_{\lambda-\frac{1}{2}}$ and $\widehat{V}' = V_{\lambda'+\frac{1}{2}}$.

[12] All the necessary facts of the quantum groups that we use below can be found in the review [64].

Using formulas (4.5) and (4.18), one now easily obtains matrix elements of the projection operator[13]:

Case A:

$$|n\rangle_{\lambda-\frac{1}{2}}|n'\rangle_{\lambda'-\frac{1}{2}} \longrightarrow q^{\frac{n'-n-1}{2}}\left([n'-2\lambda']_q q^{\lambda'}|n+1\rangle_\lambda|n'\rangle_{\lambda'} - [n-2\lambda]_q q^{-\lambda}|n\rangle_\lambda|n'+1\rangle_{\lambda'}\right) \tag{4.22}$$

Case B:

$$|n\rangle_{\lambda+\frac{1}{2}}|n'\rangle_{\lambda'-\frac{1}{2}} \longrightarrow q^{\frac{n'-n-1}{2}}\left([n'-2\lambda']_q q^{\lambda'}|n\rangle_\lambda|n'\rangle_{\lambda'} - [n]_q q^{+\lambda+1}|n-1\rangle_\lambda|n'+1\rangle_{\lambda'}\right) \tag{4.23}$$

Using explicit formulas (4.22)-(4.23) for the matrix elements of $\Gamma(l,l'|n,n')$, one can find for the identities (4.14) a simple generating equation (see the end of the previous subsection) provided the evolution operators are choosen to be q-exponential[14] of the generators $U(t)=e_q(tT_+)$, $\overline{U}(\bar{t}) = e_q(\bar{t}T_-)$. This generating equation is of the form

Case A:

$$\begin{aligned} &\sqrt{M^-_{\bar{t}}M^+_{\bar{t}'}}\left(q^{\lambda'}D^{(0)}_{\bar{t}}\bar{t}'D^{(2\lambda')}_{\bar{t}'} - q^{-\lambda}\bar{t}D^{(2\lambda)}_{\bar{t}}D^{(0)}_{\bar{t}'}\right)\tau_\lambda(t,\bar{t}|g)\tau_{\lambda'}(t',\bar{t}'|g) = \\ &= [2\lambda]_q[2\lambda']_q\sqrt{M^-_tM^+_{t'}}\left(q^{-(\lambda+\frac{1}{2})}t' - q^{(\lambda'+\frac{1}{2})}t\right)\tau_{\lambda-\frac{1}{2}}(t,\bar{t}|g)\tau_{\lambda'-\frac{1}{2}}(t',\bar{t}'|g) \end{aligned} \tag{4.24}$$

Here $D^{(\alpha)}_t \equiv \frac{q^{-\alpha}M^+_t - q^\alpha M^-_t}{(q-q^{-1})t}$, and $M^\pm$ are the multiplicative shift operators: $M^\pm_t f(t) = f(q^{\pm1}t)$.

Case B:

$$\begin{aligned} &\sqrt{M^-_{\bar{t}}M^+_{\bar{t}'}}\left(q^{\lambda'}\bar{t}'D^{(2\lambda')}_{\bar{t}'} - q^{(\lambda+1)}\bar{t}D^{(0)}_{\bar{t}'}\right)\tau_\lambda(t,\bar{t}|g)\tau_{\lambda'}(t',\bar{t}'|g) = \\ &= \frac{[2\lambda']_q}{[2\lambda+1]_q}\sqrt{M^-_tM^+_{t'}}\left(q^{\lambda'}tD^{(2\lambda+1)}_t - q^\lambda t'D^{(0)}_t\right)\tau_{\lambda+\frac{1}{2}}(t,\bar{t}|g)\tau_{\lambda'-\frac{1}{2}}(t',\bar{t}'|g) \end{aligned} \tag{4.25}$$

The classical limit of these equations is

Case A:

$$\left(2\lambda\frac{\partial}{\partial\bar{t}'} - 2\lambda'\frac{\partial}{\partial\bar{t}} + (\bar{t}'-\bar{t})\frac{\partial^2}{\partial\bar{t}\partial\bar{t}'}\right)\tau_\lambda(t,\bar{t}|g)\tau_{\lambda'}(t',\bar{t}'|g) = 4\lambda\lambda'(t'-t)\tau_{\lambda-\frac{1}{2}}(t,\bar{t}|g)\tau_{\lambda'-\frac{1}{2}}(t',\bar{t}'|g) \tag{4.26}$$

Case B:

$$\begin{aligned} &\left[(\bar{t}'-\bar{t})\frac{\partial}{\partial\bar{t}'} - 2\lambda'\right]\tau_\lambda(t,\bar{t}|g)\tau_{\lambda'}(t',\bar{t}'|g) = \\ &= \frac{2\lambda'}{2\lambda+1}\left[(t-t')\frac{\partial}{\partial t} - 2\lambda - 1\right]\tau_{\lambda+\frac{1}{2}}(t,\bar{t}|g)\tau_{\lambda'-\frac{1}{2}}(t',\bar{t}'|g) \end{aligned} \tag{4.27}$$

Thus, we obtain *different* BI that are satisfied by *the same* τ-function. In fact, it suffices to use, say, the first equation (case A) to fix the τ-function completely.

Classical limit. Before going into further details, consider the case of the classical group $SL(2)$ with τ-function satisfying the BI (4.26)-(4.27). One can construct the general solution to these

[13] In order to simplify the formulas, hereafter we omit the sign of tensor product from the notation of the states $|+\rangle \otimes |0\rangle_\lambda$ etc.

[14] $e_q(x) \equiv \sum_{n\geq0}\frac{x^n}{[n]_q!}$.

equations, however, it is even simpler to obtain τ-function immediately from the definition (4.2). Taking the representation of arbitrary spin λ, one gets the result:

$$\tau_\lambda = {}_\lambda\langle 0|e^{tT_-} g e^{\bar{t}T_+}|0\rangle_\lambda = (a + b\bar{t} + ct + dt\bar{t})^{2\lambda} \tag{4.28}$$

the group element g being parametrized by three parameters:

$$g = e^{x_+T_+} e^{x_0T_0} e^{x_-T_-} \tag{4.29}$$

and

$$a \equiv e^{\frac{1}{2}x_0} + x_+x_-e^{-\frac{1}{2}x_0}, \quad b \equiv x_+e^{-\frac{1}{2}x_0}, \quad c \equiv e^{-\frac{1}{2}x_0}x_-, \quad d \equiv e^{-\frac{1}{2}x_0} \tag{4.30}$$

i.e.

$$ad - bc = 1 \tag{4.31}$$

Let us now return to the BI. It was already mentioned that it is sufficient to look at the equation (4.26), since every solution to this equation satisfies all other BI, say, the equation (4.27), or other equations obtained with other choices of V, $\widehat{V}$ and W. The general solution (4.26) is 3-parametric and, certainly, coincides with (4.28).

Let us note that (4.28) at $\lambda = \frac{1}{2}$ (fundamental representation) has little to do with the general solution of the Liouville equation (3.12) that is associated with the $SL(2)$-reduction (see section 3). However, there is a connection between them: although the set of solutions of the Liouville equation (3.10) is far more rich, solutions (4.26) are contained in this set.

In order to understand how the set of solutions of the Liouville equation should be restricted, one needs to rewrite (4.26) (like it is usually done for Hirota equations) as a system of differential equations obtained by expanding into powers of $\epsilon = \frac{1}{2}(t - t')$ and $\bar{\epsilon} = \frac{1}{2}(\bar{t} - \bar{t}')$. For instance, at $\lambda = \lambda'$ one gets from (4.26):

$$\begin{aligned}
&\text{coefficient in front of } \epsilon: \quad \partial\tau_\lambda\bar{\partial}\tau_\lambda - \tau_\lambda\partial\bar{\partial}\tau_\lambda = 2\lambda\tau^2_{\lambda-\frac{1}{2}} \\
&\text{coefficient in front of } \bar{\epsilon}: \quad 2\lambda\tau_\lambda\bar{\partial}^2\tau_\lambda = (2\lambda - 1)(\bar{\partial}\tau_\lambda)^2 \\
&\qquad\qquad\qquad\qquad\ldots
\end{aligned} \tag{4.32}$$

If $\lambda = \frac{1}{2}$, the first of these two equations is nothing but the Liouville equation (3.10):

$$\partial\tau_{\frac{1}{2}}\bar{\partial}\tau_{\frac{1}{2}} - \tau_{\frac{1}{2}}\partial\bar{\partial}\tau_{\frac{1}{2}} = \tau_0^2 = 1 \tag{4.33}$$

while the second one

$$\bar{\partial}^2\tau_{\frac{1}{2}} = 0 \tag{4.34}$$

is the condition that strongly restricts the solutions of the Liouville equation. It implies that the two arbitrary functions $A(t)$ and $B(\bar{t})$ parametrizing these solutions are reduced to linear functions of times. In terms of Grassmannian, this means that the $SL(2)$-τ-function considered here corresponds to the embedding of the $SL(2)$ matrix into the left upper corner of the $GL(\infty)$ matrix – see section 3. Certainly, it had to be expected, since $SL(2)$ τ-function "knows" nothing of the $GL(\infty)$ group. In particular, in order to quantize the "full-fledged" Liouville equation, at first, one needs to study the $GL_q(\infty)$ system, and then its corresponding reduction.

Solutions to the quantum BI. Let us discuss now solutions to the quantum BI. The general c-number solution to the equation (4.24) can be manifestly constructed. Indeed, one can easily check that

$$\tau_\lambda = [\alpha + \frac{1}{\alpha}t\bar{t}]_q^{2\lambda} \equiv \sum_{i\geq 0} \frac{\Gamma_q(2\lambda+1)}{\Gamma_q(2\lambda+1-i)} \frac{\alpha^{2\lambda-2i}(t\bar{t})^i}{[i]_q!} \tag{4.35}$$

satisfies the equation (4.24), since

$$D_t^{(0)}[\alpha + \frac{1}{\alpha}t\bar{t}]_q^{2\lambda} = \frac{1}{\alpha}[2\lambda]_q[\alpha + \frac{1}{\alpha}t\bar{t}]_q^{2\lambda-1}\bar{t},$$
$$tD_t^{(2\lambda)}[\alpha + \frac{1}{\alpha}t\bar{t}]_q^{2\lambda} = -\alpha[2\lambda]_q[\alpha + \frac{1}{\alpha}t\bar{t}]_q^{2\lambda-1} \tag{4.36}$$

However, this gives only the one-parametric set of solutions in contrast to the classical case. The explanation of this phenomenon is due to the fact that, of all the elements of $U_q(SL(2))$, only the Cartan one has the "correct" co-multiplication law (4.11), while, in the classical case, there is the three-parametric family of the group elements (4.29) having such a co-multiplication.

Quantum τ-function. However, in the quantum case, there still exists the way to construct a three-parametric family of solutions. To this end, it suffices to consider the non-commutative τ-function[15]. Indeed, (4.1) implies that τ-function takes its values in the algebra of functions on the quantum group $SL_q(2)$, i.e. is *non-commutative* quantity. For instance, in the fundamental representation, it is equal to

$$\tau_{\frac{1}{2}} = \langle +|g|+\rangle + \bar{t}\langle +|g|-\rangle + t\langle -|g|+\rangle + t\bar{t}\langle -|g|-\rangle = a + b\bar{t} + ct + dt\bar{t} \tag{4.37}$$

where the generators a, b, c, d of the algebra of functions $A(SL_q(2))$ are the elements of the matrix

$$\mathcal{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - qbc = 1 \tag{4.38}$$

with commutation relations given by the equations $\mathcal{T}\mathcal{T}\mathcal{R} = \mathcal{R}\mathcal{T}\mathcal{T}$ [19]

$$\begin{gathered} ab = qba, \\ ac = qca, \\ bd = qdb, \\ cd = qdc, \\ bc = cb, \\ ad - da = (q - q^{-1})bc \end{gathered} \tag{4.39}$$

In order to obtain such a non-commutative τ-function from (4.2), one needs to consider g as an element of UEA given over some *non-commutative* ring instead of the complex number field. This increases the number of the group elements satisfying (4.11). This ring is exactly $A_q(SL(2))$ (see the next subsection).

In order to construct non-commutative τ-function at any representation of spin λ, one can decompose this representation into the representations of spins $\lambda - \frac{1}{2}$ and $\frac{1}{2}$ [26]:

$$\begin{aligned} {}_\lambda\langle k|g|n\rangle_\lambda = q^{-\frac{k+n}{2}} \Big[& {}_{\lambda-\frac{1}{2}}\langle k|g|n\rangle_{\lambda-\frac{1}{2}} \; \langle +|g|+\rangle + q^\lambda [n]_q \; {}_{\lambda-\frac{1}{2}}\langle k|g|n-1\rangle_{\lambda-\frac{1}{2}}\langle +|g|-\rangle + \\ & + q^\lambda [k]_q \; {}_{\lambda-\frac{1}{2}}\langle k-1|g|n\rangle_{\lambda-\frac{1}{2}}\langle -|g|+\rangle + q^{2\lambda}[k]_q[n]_q \; {}_{\lambda-\frac{1}{2}}\langle k-1|g|n-1\rangle_{\lambda-\frac{1}{2}}\langle -|g|-\rangle \Big]_q \end{aligned} \tag{4.40}$$

Recurrently applying this procedure, one gets

$$\begin{aligned} \tau_\lambda(t, \bar{t}|g) &= \tau_{\lambda-\frac{1}{2}}(q^{-\frac{1}{2}}t, q^{-\frac{1}{2}}\bar{t}|g)\tau_{\frac{1}{2}}(q^{\lambda-\frac{1}{2}}t, q^{\lambda-\frac{1}{2}}\bar{t}|g) = \\ \text{if } &\overset{\lambda \in \mathbf{Z}/2}{=} \tau_{\frac{1}{2}}(q^{\frac{1}{2}-\lambda}t, q^{\frac{1}{2}-\lambda}\bar{t}|g)\tau_{\frac{1}{2}}(q^{\frac{3}{2}-\lambda}t, q^{\frac{3}{2}-\lambda}\bar{t}|g)\dots\tau_{\frac{1}{2}}(q^{\lambda-\frac{1}{2}}t, q^{\lambda-\frac{1}{2}}\bar{t}|g) \end{aligned} \tag{4.41}$$

[15] The idea of introducing non-commutative τ-function has been also proposed in [61]. Note that there were also other very interesting attempts in the same direction [21], although I do not quite understand their relation to our approach.

4.3 Universal T-operator (group element)

General construction. Let us construct now in more explicit terms the group element given over the non-commutative ring – the algebra of functions on the quantum group. That is, we construct such an element $g \in U_q(\mathcal{G}) \otimes A(\mathbf{G})$ of the tensor product of UEA $U_q(\mathcal{G})$ and its dual algebra of functions $A(\mathbf{G})$ that

$$\Delta_U(g) = g \otimes_U g \in A(\mathbf{G}) \otimes U_q(\mathcal{G}) \otimes U_q(\mathcal{G}) \tag{4.42}$$

To construct this element [19, 65, 27, 66], we fix some basis $T^{(\alpha)}$ in $U_q(\mathcal{G})$. There exists a non-degenerated pairing between $U_q(\mathcal{G})$ and $A(\mathbf{G})$, which we denote $< ... >$. We also fix the basis $X^{(\beta)}$ in $A(\mathbf{G})$ orthogonal to $T^{(\alpha)}$ w.r.t. this pairing. Then, the sum

$$\mathbf{T} \equiv \sum_\alpha X^{(\alpha)} \otimes T^{(\alpha)} \in A(\mathbf{G}) \otimes U_q(\mathcal{G}) \tag{4.43}$$

is exactly the group element we are looking for. It is called the universal **T**-matrix (as it is intertwined by the universal $\mathcal{R}$-matrix) or the universal group element.

In order to prove that (4.43) satisfies formula (4.42) one should note that the matrices $M^{\alpha\beta}_\gamma$ and $D^\alpha_{\beta\gamma}$ giving respectively the multiplication and co-multiplication in $U_q(\mathcal{G})$

$$T^{(\alpha)} \cdot T^{(\beta)} \equiv M^{\alpha\beta}_\gamma T^{(\gamma)}, \quad \Delta(T^{(\alpha)}) \equiv D^\alpha_{\beta\gamma} T^{(\beta)} \otimes T^{(\gamma)} \tag{4.44}$$

give rise to, inversely, co-multiplication and multiplication in the dual algebra $A(\mathbf{G})$:

$$\begin{aligned} D^\alpha_{\beta\gamma} &= \left\langle \Delta(T^{(\alpha)}), X^{(\beta)} \otimes X^{(\gamma)} \right\rangle \equiv \left\langle T^{(\alpha)}, X^{(\beta)} \cdot X^{(\gamma)} \right\rangle \\ M^{\alpha\beta}_\gamma &= \left\langle T^{(\alpha)} T^{(\beta)}, X^{(\gamma)} \right\rangle = \left\langle T^{(\alpha)} \otimes T^{(\beta)}, \Delta(X^{(\gamma)}) \right\rangle \end{aligned} \tag{4.45}$$

Then,

$$\begin{aligned} \Delta_U(\mathbf{T}) &= \sum_\alpha X^{(\alpha)} \otimes \Delta_U(T^{(\alpha)}) = \sum_{\alpha,\beta,\gamma} D^\alpha_{\beta\gamma} X^{(\alpha)} \otimes T^{(\beta)} \otimes T^{(\gamma)} = \\ &= \sum_{\beta,\gamma} X^{(\beta)} X^{(\gamma)} \otimes T^{(\beta)} \otimes T^{(\gamma)} = \mathbf{T} \otimes_U \mathbf{T} \end{aligned} \tag{4.46}$$

This is the first defining property of the universal **T**-operator, which coincides with the classical one. The second property that allows one to consider **T** as an element of the "true" group is the group multiplication law $g \cdot g' = g''$ given by the map:

$$g \cdot g' \equiv \mathbf{T} \otimes_A \mathbf{T} \in A(\mathbf{G}) \otimes A(\mathbf{G}) \otimes U_q(\mathcal{G}) \longrightarrow g'' \in A(\mathbf{G}) \otimes U_q(\mathcal{G}) \tag{4.47}$$

This map is canonically given by the co-multiplication and is again the universal **T**-operator:

$$\mathbf{T} \otimes_A \mathbf{T} = \sum_{\alpha,\beta} X^{(\alpha)} \otimes X^{(\beta)} \otimes T^{(\alpha)} T^{(\beta)} = \sum_{\alpha,\beta,\gamma} M^\gamma_{\alpha,\beta} X^{(\alpha)} \otimes X^{(\beta)} \otimes T^{(\gamma)} = \sum_\alpha \Delta(X^{(\alpha)}) \otimes T^{(\alpha)} \tag{4.48}$$

i.e.

$$\begin{gathered} g \equiv \mathbf{T}(X, T), \quad g' \equiv \mathbf{T}(X', T), \quad g'' \equiv \mathbf{T}(X'', T) \\ X \equiv \{X^{(\alpha)} \otimes I\} \in A(\mathbf{G}) \otimes I, \quad X' \equiv \{I \otimes X^{(\alpha)}\} \in I \otimes A(\mathbf{G}) \\ X'' \equiv \{\Delta(X^{(\alpha)})\} \in A(\mathbf{G}) \otimes A(\mathbf{G}) \end{gathered} \tag{4.49}$$

T-operator for $SL_q(2)$. In order to get more compact formulas, let us redefine the generators of $U_q(SL(2))$ so that the co-multiplication becomes non-symmetric:

$$
\begin{gathered}
T_+ \longrightarrow T_+ q^{-T_0}, \quad T_- \longrightarrow q^{T_0} T_- \\
\Delta(T_+) = I \otimes T_+ + T_+ \otimes q^{-2T_0}, \quad \Delta(T_-) = T_- \otimes I + q^{2T_0} \otimes T_-
\end{gathered}
\tag{4.50}
$$

This replace results in substituting everywhere the q-number $[n]_q$ for the q-number $(n)_q \equiv \frac{1-q^{2n}}{1-q^2}$, and the q-exponential $e_q(x)$ – for $\mathcal{E}_q(x) \equiv \frac{1}{e_q(-x)} = \sum_{k\geq 0} \frac{x^k}{[k]_q!} q^{-k(k-1)/2} = \sum_{k\geq 0} \frac{x^k}{(k)_q!}$. Besides, the difference operators $D_t^{(\alpha)} = \frac{q^{-\alpha}M_t^+ - q^{\alpha}M_t^-}{(q-q^{-1})t}$ are replaced by $D_t^{(\alpha)} = \frac{q^{2\alpha}M_t^+ - 1}{(q^2-1)t}$.

Fix now the basis $T^{(\alpha)} = T_+^i T_0^j T_-^k$ in $U_q(SL(2))$. Then, using the co-multiplication for $T^{(\alpha)}$, one can calculate the matrix $D^{\alpha}_{\beta\gamma}$ (4.45) and construct manifest the orthonormal basis $X^{(\alpha)}$:

$$
X^{(\alpha)} = \frac{x_+^i}{(i)_q!} \frac{x_0^j}{j!} \frac{x_-^k}{(k)_{q^{-1}}!}
\tag{4.51}
$$

where generating elements $x_\pm,\ x_0$ produce the Borel Lie algebra

$$
[x_0, x_\pm] = (\log q) x_\pm, \quad [x_+, x_-] = 0
\tag{4.52}
$$

Thus,

$$
\mathbf{T} = \mathcal{E}_q^{x_+ T_+} e^{x_0 T_0} \mathcal{E}_{q^{-1}}^{x_- T_-}
\tag{4.53}
$$

General formula for T-operator. In the case of general (simple) Lie algebra, there are different explicit representations of the $\mathbf{T}$-operator. For instance, in [65], in accordance with the general construction (4.43), there was considered the so-called PBW-basis (Poincare-Birkgoff-Witt basis) [64], presented by the ordered monomials of the algebra generators corresponding to all roots. In this basis, the group element is represented by the product of the q-exponentials like (4.53) given by the Gauss decomposition, and the generating elements of the algebra of functions satisfy the relations like (4.52). In fact, these relations give again a Borel Lie algebra. It is related to the general structure of the algebra of functions as a double (see [27] and reference [11] therein).

Another interesting representation for the group element of arbitrary simple Lie algebra has been found in [66]. This representation is remarkable, since it is constructed in the Chevalle basis of UEA, i.e. in the generators corresponding to the simple roots. To construct the linear basis in UEA in terms of Chevalle generators for arbitrary simple Lie algebra is a non-trivial problem, since they satisfy some additional constraints – Serre relations [67], and, therefore, are not free generating elements. However, in the paper [66], the group element was constructed in these terms not basing on expressions like (4.43), but by the immediate check of the relation (4.42) taking into account the Serre relations.

Namely, in [66] the group element has been presented in the form:

$$
\begin{gathered}
g = g_U g_D g_L \\
g_U = \prod_s^{<} \mathcal{E}_q^{\theta_s T_{i(s)}}, \quad g_L = \prod_s^{>} \mathcal{E}_{q^{-1}}^{\chi_s T_{-i(s)}}, \quad g_D = \prod_{i=1}^{r_{\mathbf{G}}} e^{\vec{\phi}\vec{H}}
\end{gathered}
\tag{4.54}
$$

where generators T_i correspond only to the *simple* roots $\pm\vec{\alpha}_i$, $i = 1, \ldots, r_{\mathbf{G}}$. Every root $\vec{\alpha}_i$ may appear some times in this product so that there are different parametrizations of the group element, which depend on the choice of the set $\{s\}$ and of the map $i(s)$ of the choosen set to the set of the simple roots. Technically, the calculations in terms of Chevalle generators are easier because of the specifically simple co-multiplication law:

$$
\begin{gathered}
\Delta(T_i) = T_i \otimes q^{-2H_i} + I \otimes T_i \\
\Delta(T_{-i}) = T_{-i} \otimes I + q^{2H_i} \otimes T_{-i}
\end{gathered}
\tag{4.55}
$$

Note that formula (4.54) is written down for the simple-laced algebras. Generally, one needs to consider the q-exponential with parameter $q^{||\vec{\alpha}_i||^2/2}$, not q.

The generating elements of the algebra of functions $\theta, \chi, \vec{\phi}$ in this parametrization satisfy the quadratic algebra that is the exponential of the Heisenberg algebra:

$$
\begin{aligned}
\theta_s \theta_{s'} &= q^{-\vec{\alpha}_{i(s)}\vec{\alpha}_{i(s')}}\theta_{s'}\theta_s, \quad s < s' \\
\chi_s \chi_{s'} &= q^{-\vec{\alpha}_{i(s)}\vec{\alpha}_{i(s')}}\chi_{s'}\chi_s, \quad s < s' \\
e^{\vec{\beta}\vec{\phi}}\theta_s &= q^{\vec{\beta}\vec{\alpha}_{i(s)}}\theta_s e^{\vec{\beta}\vec{\phi}} \\
e^{\vec{\beta}\vec{\phi}}\chi_s &= q^{\vec{\beta}\vec{\alpha}_{i(s)}}\chi_s e^{\vec{\beta}\vec{\phi}}
\end{aligned}
\tag{4.56}
$$

These relations can be read off from the parametrization (4.54) and formula (4.42).

4.4 τ-function and representations of the algebra of functions

In accordance with our definition, the non-commutative τ-function is an element of the algebra of functions on the quantum group. Therefore, the natural problem is, having some fixed representation of this algebra, to determine the "value" of the τ-function in this representation. At the same time, each representation has to be associated with some solution of the integrable hierarchy. In fact, this is just invariant description for the group acting in the space of the spectral parameter [31]. Indeed, any concrete solution of the classical 2DTL hierarchy is described by the τ-function (2.1) with some concrete matrix $\mathcal{G}_{mn}$ that gives the element of the Grassmannian (2.6). In terms of the group element, this means that we fix some trivial representation of the algebra of functions merely given by c-numbers. These representations exhaust the representations of the algebra on the classical group, however, in the quantum case, there are non-trivial representations. Nevertheless, one still should identify every representation of $A(\mathbf{G})$ with some solution of the BI hierarchy. Certainly, any reduction of the integrable system selects some subspace in the solution space, i.e. restricts the class of representations under consideration, and can be described, as a rule, by some additional group structure (similar to the Toda molecule – see section 3).

Thus, we come to the following general algebraic scheme of constructing integrable hierarchy:

For any given UEA $U(\mathcal{G})$, using the procedure described in the present section, one can introduce the τ-function that satisfies BI and takes its values in the algebra $A(\mathbf{G})$ of functions on the group. Besides, a natural action of UEA on the algebra of functions is fixed, and any concrete solution of the BI, in the case of the Lie algebra corresponding to the solution of the classical hierarchy, is given by fixing the representation of the algebra of functions.

Let us consider now the quantum τ-function that is an operator. Then, the natural question is whether one is able to make some c-number quantities of it. This is important, say, to expose the connections between τ-function and the generating function of correlators in quantum system. The simplest c-number quantity is the "double" generating function that generates matrix elements of both the representation and co-representation of UEA (any co-representation of UEA is equivalent, by duality, to some representation of the algebra of functions). This generating function depends on four sets of times and should satisfy BI w.r.t. the indices of both the representation and the co-representation, i.e. describe some *four-dimensional* equation system.

Another c-number function is the τ-function itself taken in the trivial co-representation. This case is closest to the classical case and, hence, is especially interesting. We discuss this case in detail later, however, first, we describe briefly the structure of the co-representations of quantum groups. We consider the simplest case of $SL_q(2)$ (more general consideration – see [27] and reference [11] therein).

Co-representations of UEA $SL_q(2)$ are given by the representations of the algebra (4.52). This is Borel algebra, therefore, it has no non-trivial finite-dimensional irreducible representations [67]. Its finite-dimensional representations are reducible but not completely reducible. The irreducible representations have been considered first in [68][16]. To compare with the results of this paper, rewrite the standard generators of $A(SL_q(2))$ (4.39) in terms of the algebra (4.52) (this is a sort of bosonization of the algebra of functions):

$$a = e^{\frac{1}{2}x_0} + x_+x_-e^{-\frac{1}{2}x_0}, \quad b = x_+e^{-\frac{1}{2}x_0}, \quad c = e^{-\frac{1}{2}x_0}x_-, \quad d = e^{-\frac{1}{2}x_0} \tag{4.57}$$

It is remarkable that these expressions coincide with (4.30), but different x are not commuting.

In these terms, the only two irreducible representations are the trivial one given by $x_+ = x_- = 0$, i.e. $ad = 1,\ b = c = 0$, and the infinite-dimensional representation that can be given explicitly by the action on the basis $\{e_k\}_{k\geq 0}$ [68]

$$ae_k = (1-q^{2k})^{\frac{1}{2}}e_{k-1}\ (ae_0 = 0), \quad de_k = (1-q^{2k+2})^{\frac{1}{2}}e_{k+1}, \quad ce_k = \theta q^k e_k, \quad be_k = -\theta^{-1}q^{k+1}e_k \tag{4.58}$$

This structure of representations can be easily generalized to other quantum groups (of rank r), since the algebras of the generating elements x are always Borel ones. Thus, the whole set of irreducible representations can be again exhausted by the trivial and infinite-dimensional representations (all of them being r-parametric). The explicit formulas for them can be found in [69] (see also [27] and reference [11] therein).

τ-function in the trivial representation and difference KOS hierarchy. Thus, τ-function in the trivial representation gives some pattern of the c-number function. This function is, certainly, too simple and flat. However, now we obtain the equation satisfied by the τ-function in the trivial representation, which has as its solutions quite non-trivial c-number functions.

First note that, even in the case of $SL_q(2)$, there is no naive determinant representation similar to (2.55). Indeed, introduce (for the sake of brevity, we denote $D \equiv D^{(1)}$):

$$C_1^1 = \tau_F = a + b\bar{t} + ct + dt\bar{t} \tag{4.59}$$

Then,

$$C_2^1 = D_{\bar{t}}C_1^1 = b + dt, \quad C_1^2 = D_tC_1^1 = c + d\bar{t}, \quad C_2^2 = D_{\bar{t}}D_tC_1^1 = d \tag{4.60}$$

and C_b^a can not be identified with the generating elements of the coordinate ring $SL_q(2)$ a, b, c, d (4.39) (for instance, $C_2^1C_1^2 \neq C_1^2C_2^1$). Thus, the determinant $\det_q C$ is not the adequate object and, hence, it (either the definition of C) should be modified. Namely, in the case of $SL_q(2)$, the appropriate formula is of the form

$$\tau_{F^{(2)}} = \det_q g = 1 = C_1^1C_2^2 - qC_2^1M_{\bar{t}}^-C_1^2 = \tau_FD_tD_{\bar{t}}\tau_F - qD_{\bar{t}}\tau_FM_t^-D_t\tau_F \tag{4.61}$$

We will return to the determinant formulas for higher groups in the next section. Now let us note that the τ-function in the trivial representation $A(SL_q(2)) - ad = 1,\ b = c = 0$ satisfies the simpler equation (it has been first proposed in [70], and we abbreviate it as KOS, in accordance with the first letters of the names of the authors)

$$\tau_FD_tD_{\bar{t}}\tau_F - D_{\bar{t}}\tau_FD_t\tau_F = 1 \tag{4.62}$$

This equation is of the form close to the equation (3.4), which is easily extendable to the $SL_q(n)$-case, in contrast to (4.61). This is not surprising, since the equation (4.62) is solved by the τ-function in the trivial representation.

[16] Striktly speaking, in [68] there have been discussed the $*$-representations, i.e. the representations with an additional involution.

One more argument in favor of the equation (4.62) is that it naturally leads to the determinant of the Schur q-polynomials (4.65). Indeed, consider the trivial element g (corresponding to the trivial representation of $A(\mathbf{G})$). Then, for the case of $SL_q(n)$, one can get by the direct calculation (see [27] and reference [5] therein)

$$\langle k,0,\dots,0|\mathcal{E}_{q^{-1}}^{s_2T_{12}}\mathcal{E}_{q^{-1}}^{s_3T_{13}}\dots\mathcal{E}_{q^{-1}}^{s_nT_{1n}}\times\mathcal{E}_{q}^{\bar{s}_2T_{21}}\mathcal{E}_{q}^{\bar{s}_3T_{31}}\dots\mathcal{E}_{q}^{\bar{s}_nT_{n1}}|k,0,\dots,0\rangle = P_k^{(q)}(s\bar{s}) \tag{4.63}$$

where $\langle k,0,\dots,0|$ is the symmetric product of the k simplest fundamental representations.

4.5 Difference KOS hierarchy

Difference KOS hierarchy from the 2DTL hierarchy. Consider the equation (4.62) more carefully and, in particular, demonstrate that it can be obtained as the equation of the classical 2DTL hierarchy with a non-trivial evolution [25], i.e. that the *difference* equation (KOS) (4.62) can be obtained in the framework of the standard *differential* ($GL(\infty)$) 2DTL hierarchy. To this end, let us consider the τ-function[17], defined just by (2.24) with redefined time flows

$$C_l^k(s,\bar{s}) \longrightarrow \mathcal{C}_l^k(s,\bar{s}) = \sum_{i,j} P_{i-k}^{(q)}(s)\ R_{ij}\ P_{j-l}^{(q)}(\bar{s}) \tag{4.64}$$

where Schur q-polynomials are defined by the formula

$$\prod_i \mathcal{E}_{q^i}(s_i z^i) = \sum_j P_j^{(q)}(s) z^j \tag{4.65}$$

and satisfy the conditions

$$D_{s_i}P_j^{(q)}(s) = (D_{s_1})^i P_j^{(q)}(s) = P_{j-i}^{(q)}(s) \tag{4.66}$$

Thus, we obtain

$$D_{s_i}\mathcal{C}_l^k = \mathcal{C}_l^{k+i},\quad D_{\bar{s}_i}\mathcal{C}_l^k = \mathcal{C}_{l+i}^k \tag{4.67}$$

and

$$\tau_n^{(P^{(q)})}(s,\bar{s}|g) = \det{}_{1\le k,l\le n} D_{s_1}^{k-1}D_{\bar{s}_1}^{l-1}\mathcal{C}_1^1(s,\bar{s}) \tag{4.68}$$

Thus defined τ-function, indeed, satisfies the equations (4.62) [70, 25]:

$$\begin{gathered}\tau_k\cdot D_{s_1}D_{\bar{s}_1}\tau_k - D_{s_1}\tau_k\cdot D_{\bar{s}_1}\tau_k = \tau_{k-1}\cdot M_{s_1}^{+}M_{\bar{s}_1}^{+}\tau_{k+1}\\ \dots\end{gathered} \tag{4.69}$$

where ellipses means the rest of equations of the KOS hierarchy. The simplest way to prove this equation is to rewrite the τ-function using the formula

$$\det D_{s_1}^i D_{\bar{s}_1}^j C = q^{-(n-1)(n-2)}(1-q)^{n(n-1)}(t\bar{t})^{\frac{n(n-1)}{2}}\det{}_{0\le i,j<n}(M_{s_1}^{+})^i(M_{\bar{s}_1}^{+})^j C \tag{4.70}$$

and then apply the Jacobi identity[18].

---

[17] In the next section, we show that this τ-function is the generating function of the matrix elements in fundamental representations.

[18] The Jacobi identity is the particular ($p=2$) case of the general identity for the minors of arbitrary matrix

$$\sum_{i_p} C_{ri_p}\widehat{C}_{i_1\dots i_p|j_1\dots j_p} = \frac{1}{p!}\sum_P(-)^P\widehat{C}_{i_1\dots i_{p-1}|j_{P(1)}\dots j_{P(p-1)}}\delta_{rj_{P(p)}}$$

where the sum in the r.h.s. goes over all permutations of p indices, and $\widehat{C}_{i_1\dots i_p|j_1\dots j_p}$ denote the determinant (minor) of the matrix obtained from C_{ij} by removing rows $i_1\dots i_p$ and columns $j_1\dots j_p$. Using that $(C^{-1})_{ij} = \widehat{C}_{i|j}/\widehat{C}$, one can rewrite this identity as

$$\widehat{C}\widehat{C}_{i_1\dots i_p|j_1\dots j_p} = \left(\frac{1}{p!}\right)^2\sum_{P,P'}(-)^P(-)^{P'}\widehat{C}_{i_{P'(1)}\dots i_{P'(p-1)}|j_{P(1)}\dots j_{P(p-1)}}\delta_{i_{P'(p)}|j_{P(p)}}$$

Fermionic language for the KOS hierarchy. To conclude this section, we show how the KOS hierarchy can be rewritten in the fermionic language. Since the τ-function is obtained from the Toda hierarchy by the redefinition of times, one can simply substitute the new times into the τ-function of the 2DTL hierarchy. Indeed, the relation

$$\prod_{k=1}^{\infty} \mathcal{E}_{q^k}(s_k z^k) = \prod_{k=1}^{\infty} e^{t_k z^k} \tag{4.71}$$

allows one to express times t through s

$$\sum_{k=1}^{\infty} t_k z^k = \sum_{n,k=1}^{\infty} \frac{s_k^n (1-q_k)^n}{n(1-q_k^n)} z^{nk} \tag{4.72}$$

so that

$$P_k^{(q)}(s) = P_k(t) \tag{4.73}$$

Thus, the τ-function can be presented in the form

$$\tau_n(s,\bar{s}|g) = \tau_n(t,\bar{t}|g) \overset{(2.1)}{\sim} \langle n| e^{H(t)} g e^{\overline{H}(\bar{t})} |n\rangle \tag{4.74}$$

where

$$\begin{aligned} H(t) &= \sum_{n>0} t_n J_{+n} \overset{(4.72)}{=} \sum_{n,k=0}^{\infty} \frac{s_k^n (1-q_k)^n}{n(1-q_k^n)} J_{+nk} \\ \overline{H}(\bar{t}) &= \sum_{n>0} \bar{t}_n J_{-n} = \sum_{n,k=0}^{\infty} \frac{\bar{s}_k^n (1-q_k)^n}{n(1-q_k^n)} J_{-nk} \end{aligned} \tag{4.75}$$

τ-function of the KOS hierarchy can be also considered as the τ-function of the 2DTL hierarchy in Miwa variables [25]. To this end, one needs to consider the general Miwa transform in some special points. That is, using formulas like

$$t_k = \frac{1}{k} \frac{((1-q)s_1)^k}{1-q^k} = \frac{1}{k} \sum_{l\geq 0} \Big((1-q) q^l s_1\Big)^k \tag{4.76}$$

one easily gets that the τ-function (4.74) is described by the following set of the Miwa variables

$$\Big\{ e^{2\pi i a/k} \mu_k q_k^{-l/k} \Big| \, a = 0, \dots k-1; \;\; l \geq 0 \Big\}, \quad \mu_k = ((1-q_k)s_k)^{-1/k} \tag{4.77}$$

This means that the KOS hierarchy can be considered as the 2DTL hierarchy in Miwa variables with the special choice (4.77) of these latter.

5 Quantum and classical KP hierarchy with different evolutions

5.1 Structure of fundamental representations

In this section, we investigate a particular case of the construction considered above, that is, the integrable hierarchies associated with the fundamental representations of the groups $SL(p)$ and $SL_q(p)$ [28, 29, 26, 27]. They are of great importance, since it is these cases that correspond to the standard 2DTL hierarchy and its quantum counterpart and, besides, just in these cases there are some determinant representations for the τ-functions. Note that, in order to obtain determinant

formulas in quantum case, we have to extend slightly the definition of the τ-function (4.2) by introducing non-commutative times.

Strictly speaking, the 2DTL hierarchy is described by the group $SL(\infty)$. However, we consider the case of general p, although the results almost do not depend on p.

We start with describing the fundamental representations of these groups [67] (see also Appendix 2). The group $SL(p)$ has $r \equiv \text{rank } = p-1$ fundamental representations, the simplest one $F \equiv F_1$ being the p-plet containing the states

$$\psi_i = T_-^{i-1}|0\rangle, \quad i = 1, \ldots, p \tag{5.1}$$

Here the generator T_- is the sum of all r *simple* roots of $SL(p)$: $T_- = \sum_{i=1}^r T_{-\alpha_i}$. All other fundamental representations F_k can be now constructed as skew degrees of $F = F_1$:

$$F^{(k)} = \left\{ \Psi^{(k)}_{i_1 \ldots i_k} \sim \psi_{[i_1} \ldots \psi_{i_k]} \right\} \tag{5.2}$$

F_k is given by action of the operators

$$R_k(T_-^i) \equiv T_-^i \otimes I \otimes \ldots \otimes I + I \otimes T_-^i \otimes \ldots \otimes I + I \otimes I \otimes \ldots \otimes T_-^i \tag{5.3}$$

on the highest weight vector. These operators commute with each other. Given an integer k, evidently, there are exactly k independent ones among them (with $i = 1, \ldots, k$). The fact that all the fundamental representations are generated by the same generator T_- is a remarkable property that can serve as a definition of the fundamental representations and is in charge of all essential features of the classical integrable hierarchies, in particular, of many commuting Hamiltonians.

Now one can manifestly construct either a pair of the intertwining operators analogous to those in s.4.2:

$$\begin{gathered} I_{(k)}: \quad F_{k+1} \longrightarrow F_k \otimes F, \quad I^*_{(k)}: \quad F_{k-1} \longrightarrow F^* \otimes F_k \\ \text{so that} \quad \Gamma_{k|k'}: \quad F_{k+1} \otimes F_{k'-1} \longrightarrow F_k \otimes F_{k'} \end{gathered} \tag{5.4}$$

or a pair of the fermionic intertwining operator

$$\psi^+: \quad F_1 \otimes F_k \longrightarrow F_{k+1}, \quad \psi^- \equiv I_{(k)} \tag{5.5}$$

Here

$$\begin{gathered} F^* = F^{(r)} = \left\{ \psi^i \sim \epsilon^{i i_1 \ldots i_r} \psi_{[i_1} \ldots \psi_{i_r]} \right\} \\ I_{(k)}: \quad \Psi^{(k+1)}_{i_1 \ldots i_{k+1}} = \Psi^{(k)}_{[i_1 \ldots i_k} \psi_{i_{k+1}]}, \quad I^*_{(k)}: \quad \Psi^{(k-1)}_{i_1 \ldots i_{k-1}} = \Psi^{(k)}_{i_1 \ldots i_{k-1} i} \psi^i \end{gathered} \tag{5.6}$$

and $\Gamma_{k|k'}$ is constructed through the embedding $I \longrightarrow F \otimes F^*$ induced by the pairing $\psi_i \psi^i$: the basis in the linear space $F^{(k+1)} \otimes F^{(k'-1)}$ induced by $\Gamma_{k|k'}$ from the basis in the space $F^{(k)} \otimes F^{(k')}$ is $\Psi^{(k)}_{[i_1 \ldots i_k} \Psi^{(k')}_{i_{k+1}] i'_1 \ldots i'_{k'-1}}$.

Now one can rewrite the operator Γ in terms of matrix elements

$$g^{(k)} \begin{pmatrix} i_1 \ldots i_k \\ j_1 \ldots j_k \end{pmatrix} \equiv \langle \Psi_{i_1 \ldots i_k} | g | \Psi_{j_1 \ldots j_k} \rangle = \det_{1 \le a,b \le k} g^{i_a}_{j_b} \tag{5.7}$$

in the following way

$$g^{(k)} \begin{pmatrix} i_1 \ldots i_k \\ [j_1 \ldots j_k \end{pmatrix} g^{(k')} \begin{pmatrix} i'_1 \ldots i'_k \\ j_{k+1}] j'_1 \ldots j'_{k'-1} \end{pmatrix} = g^{(k+1)} \begin{pmatrix} i_1 \ldots i_k [i'_{k'} \\ j_1 \ldots j_{k+1} \end{pmatrix} g^{(k'-1)} \begin{pmatrix} i'_1 \ldots i'_{k'-1}] \\ j'_1 \ldots j'_{k'-1} \end{pmatrix} \tag{5.8}$$

This is the manifest form of BI (4.10) for matrix elements in the case of fundamental representations, which is identically hold for any $g^{(k)}$ of the form (5.7). Making use of "dressing" the matrix

elements of $g^{(k)}$ by the Schur polynomials depending on times, one can easily get (2.24) from (5.7) and the 2DTL equation (3.4) – from (5.8). However, later on we always work with BI in UEA and use the fermionic pair of the intertwining operators (5.5).

Note that the fundamental representations for the quantum group $SL_q(p)$ have the same structure with antisymmetrization replaced by q-antisymmetrization. Some details can be found in Appendix 2 and [26].

Let us now turn to more invariant presentation. Namely, note that, as described above, the fundamental representations are completely generated by the operators $T_\pm^{(k)} = \sum_{\vec\alpha:h(\vec\alpha)=k} T_{\pm\vec\alpha}$ that are sums of all the generators of $SL(p)$, associated with the positive/negative roots of "weight" k (in the first fundamental representation F, these $T_\pm^{(k)}$ are the $p\times p$ matrices with units on the k-th upper/lower diagonal and zeroes wherever else.

It was already mentioned in s.4.1 that the evolution operators $U(t)$, $\overline{U}(\bar t)$ in general representation should contain *all* raising, respectively, lowering operators[19] and correspond to non-commutative flows. At the same time, for the fundamental representations F_n they can be choosen in the form

$$U(t) = \exp\left(\sum_{k\geq 1} t_k T_+^{(k)}\right),$$
$$\overline{U}(\bar t) = \exp\left(\sum_{k\geq 1} \bar t_k T_-^{(k)}\right) \tag{5.9}$$

The essential property of the operators $T_\pm^{(k)}$ is that they are commuting:

$$\left[T_+^{(k)},\ T_+^{(l)}\right] = 0, \quad \left[T_-^{(k)},\ T_-^{(l)}\right] = 0 \tag{5.10}$$

hence, $U(t)$, $\overline{U}(\bar t)$ (5.9) correspond to commuting flows. Certainly, general evolution operators depend on additional times and mutually non-commutative generators of the group.

The operators defined in (5.9) celebrate the following properties:

(i) $U,\ \overline{U}\ \in\ SL(p)$;

(ii) more concretely, U, $\overline{U}$ belong to the nilpotent subgroup $NSL(p)$ of the group $SL(p)$. In fact, $NSL(p)$ is a subgroup of the Borel subgroup: $NSL(p)\subset BSL(p)\subset SL(p)$ (in the fundamental representation F_1, the subgroup $BSL(p)$ consists of all the upper-triangle matrices with unit determinant, while the matrices from $NSL(p)$ are additionally constrained to have only units on the main diagonal);

(iii) since co-multiplication is

$$\Delta(T_{\pm\vec\alpha}) = T_{\pm\vec\alpha}\otimes I + I\otimes T_{\pm\vec\alpha} \tag{5.11}$$

then

$$\Delta U(t) = U(t)\otimes U(t) = (U(t)\otimes I)\,(I\otimes U(t)) \tag{5.12}$$

In other words, the evolution operators U, $\overline{U}$ are the group elements of the subgroup $NSL(p)$.

These properties are rather appealing and it is natural to try to preserve them in quantization. However, there are two immediate things to be taken into account. First, there is nothing similar to the operators $T_\pm^{(k)}$ at $q\neq 1$ (at least, formulated independently of a specific representation R). This means that the manifest expressions for $U(t)$ and $\overline{U}(\bar t)$ should be very different from (5.9).

Second, there is no reasonable notion of the nilpotent subgroup $N\mathbf{G}_q$ in quantum case: only the quantum deformation of the Borel subgroup $B\mathbf{G}_q\subset\mathbf{G}_q$ is well defined (see s.4.3). Indeed, if

[19] Since τ-function is, by definition, the generating function of *all* the matrix elements.

one chooses as $U(t)$ an object like g_U and as $\overline{U}(\overline{t})$ – like g_L (see (4.54)), due to the absence of the factor g_D, $\Delta(g_U) \neq g_U \otimes g_U$, $\Delta(g_L) \neq g_L \otimes g_L$, and the nilpotent subgroup $N\mathbf{G}_q$ is really absent. At the same time, the Borel subgroup $B\mathbf{G}_q$ exists, since $\Delta(g_U g_D) = (g_U g_D) \otimes (g_U g_D))$.

Despite this problem, we consider U and $\overline{U}$ as quantities like g_U and g_L respectively (however, see the end of this section). By this reason, formula (5.12) in the quantum case requires some modification. More precisely, instead of (5.12) we obtain

$$\Delta(U(\xi)) = U_L^{(2)}(\xi) \cdot U_R^{(2)}(\xi) \tag{5.13}$$

where

$$U(\xi) = \prod_s^{<} \mathcal{E}_q\left(\xi_s T_{i(s)}\right) \tag{5.14}$$

$$\begin{aligned} U_L^{(2)} &= \prod_s^{<} \mathcal{E}_q\left(\xi_s T_{i(s)} \otimes q^{-2H_{i(s)}}\right) \neq I \otimes U(\xi) \\ U_R^{(2)} &= \prod_s^{<} \mathcal{E}_q\left(\xi_s I \otimes T_{i(s)}\right) = I \otimes U(\xi) \end{aligned} \tag{5.15}$$

This expression is later used to derive the determinant formulas for the quantum τ-function.

Note that, taking the evolution operators as elements of the decomposition (4.54), we admit the new approach to the generalized τ-function when not only the group element but also the operators U and $\overline{U}$ are elements of $U(\mathcal{G}) \otimes A(\mathbf{G})$, i.e. times are *non-commutative* parameters. Exactly this approach allows one to construct determinant representations for the τ-functions.

Let us point out that, hereafter within this framework, we always understand by multiplication of the evolution operators U and $\overline{U}$ and group element g in the definition of the τ-function (4.2) the group multiplication law (4.47), that is to say, the elements θ, ϕ and χ of the algebra (4.56) in evolution operators commute with elements of the corresponding algebra in g, see s.4.3.

5.2 Group element parametrization in fundamental representations

Now let us return to the general parametrization (4.54) of the group element given in terms of Chevalle generators and consider the group $SL(p)$. Then, the simplest way to choose the map $i(s)$ is

$$i(s): \quad 1,2,\dots,r-1,r;\ 1,2,\dots,r-1;\ 1,2,3;\ 1,2;\ 1$$

with $s = 1,\dots,\frac{p(p-1)}{2}$ (dimension of the group $SL(p)$), i.e.

$$U(\xi) = \prod_{1\le i\le p} \prod_{i<j\le p} \exp\left(\xi_{ij} T_{j-i}\right) \tag{5.16}$$

However, since we are going to discuss the fundamental representation, it suffices, how one could see in the previous subsection, to consider the orbits of $SL(p)$ parametrized by only r variables. Therefore, out next goal is to find an adequate parametrization of these orbits. In the classical case ($q = 1$), there exist, at least, three such parametrizations considered below [28, 29]. However, only one of them admits a simple quantum deformation but, instead, does not correspond to (5.9). The problem with constructing these orbits of smaller dimensions is due to necessity of the reduction consistent with the commutation relations (4.56), i.e. due to necessity of choosing a subclass of rather specific representations of the algebra of functions on quantum group.

Parametrization A. The simplest possibility is to restrict the set $\{s\}$ onto $s = 1, \dots, r$ and choose $i(s) = s$, i.e.

$$U^{(A)}(\xi) = \prod_{i=1}^{r_{\mathbf{G}}\ <} \exp\left(\xi_i T_i\right) \tag{5.17}$$

This is sufficient to generate all the states of any fundamental representation from the corresponding vacuum vector (highest weight vector) but $< 0_{F_n}|\ U^{(A)}(\xi)$ has little to do with $< 0_{F_n}|\ U(t)$ (with $U(t)$ given by formula (5.9)). Maybe it is better to say that the identification $< 0_{F_n}|\ U^{(A)}(\xi) =< 0_{F_n}|\ U(t)$ gives rise to a quite complicated map $\xi_i(t)$ that manifestly depends on p.

One can certainly construct the KP/Toda hierarchy in terms of variables ξ instead of the usual times t (see below) but it is *not* obtained just as a replace of variables – the whole construction looks absolutely different. This is the price for easy generalization of this construction to the case of $q \neq 1$: one needs just to write instead of (5.17)

$$U^{(A)}(\xi) = \prod_{i=1}^{r_{\mathbf{G}}\ <} \mathcal{E}_q\left(\xi_i T_i\right) \tag{5.18}$$

where ξ are non-commuting quantities,

$$\xi_i \xi_j = q^{-\vec{\alpha}_i \vec{\alpha}_j} \xi_j \xi_i, \quad i < j \tag{5.19}$$

It is shown below that now any statement at $q = 1$ in ξ-parametrization is easily generalized to the case of $q \neq 1$.

Parametrization B (conventional). The other choice of parametrization is related to the conventional times of the KP/Toda hierarchy:

$$U^{(B)}(\xi) = \prod_s \exp\left(\xi_s T_{i(s)}\right) = U(t) = \exp\left(\sum_k t_k T_+^{(k)}\right) \tag{5.20}$$

This implies that $< 0_{F_n}|\ U^{(B)}(\xi) =< 0_{F_n}|\ U(t)$ with some p-independent functions $\xi_s(t)$. The key difference between the two sides of the equality (5.20) is that the r.h.s. contains mutually-commuting combinations of root generators, while the l.h.s. contains only mutually non-commuting Chevalle generators. Such a reparametrization actually exists, but the set $\{s\}$ should contain, at least, $\frac{p(p-1)}{2}$ elements and one can take $i(s)$ just as in (5.16). However, now not all of the ξ_s's are independent: instead they are expressed through r times t_k. For instance, the t_1-dependence of ξ_{ij} is given by

$$\xi_{ij} = \frac{t_1}{p + i - j} + \mathcal{O}(t_2, t_3, \dots) \tag{5.21}$$

However, in order to construct some reasonable quantum deformation of parametrization B, one needs to reproduce the proper commutation relations

$$\xi_s \xi_{s'} = q^{-\vec{\alpha}_{i(s)} \vec{\alpha}_{i(s')}} \xi_{s'} \xi_s, \quad s < s' \tag{5.22}$$

for $\frac{p(p-1)}{2}$ variables ξ_s as a corollary of *some* relations between r variables t_k (which, of course, do not commute when $q \neq 1$). To make this possible, one should also somehow deform formulas (5.21) at $q \neq 1$. Solution to this problem is unknown so far.

Parametrization C (Miwa variables). Yet another option is to make the (representation-independent) Miwa transform $t_k = \frac{1}{k}\sum_a \lambda_a^k$. This transformation is perfectly consistent with the simple-root decomposition:

$$U(t) = \prod_a \exp\left(\sum_{k=1}^{r_{\mathbf{G}}} \frac{\lambda_a^k}{k} T_+^{(k)}\right) = \prod_a \left(\prod_{i=1}^{r_{\mathbf{G}}} e^{\lambda_a T_i}\right) \tag{5.23}$$

The set $\{s\}$ and mapping $i(s)$ are not of the "most economic" type (5.16), but the general rule (4.56) of the quantum deformation is, of course, applicable.

In this parametrization, however, the problem is that (5.16) implies the quantum formula in the form

$$\prod_a \left(\prod_{i=1}^{r_{\mathbf{G}}} \mathcal{E}_q\left(\lambda_{ai} T_i\right)\right) \tag{5.24}$$

where λ_{ai} with different i and the same a do not commute. At the same time, the constraint $\lambda_{ai} = \lambda_{aj}$ for $i \neq j$ is of crucial importance for the classical ($q = 1$) formula (5.23). What is the proper quantum deformation of this constraint remains unclear.

5.3 Classical ($q = 1$) KP/Toda hierarchy in different parametrizations

Determinant formulas and system of equations. To understand better the scheme developed here, let us rederive now the standard features of classical integrable hierarchies within the group theory framework under consideration. These features include determinant representations for the τ-functions and the set of differential BI. We consider the fundamental representations of the group $SL(p)$, since just in this case there exist determinant representations for the τ-functions. For their derivation, one does not need to specify any parametrization and can consider some arbitrary evolution $U(t)$.

Thus, we consider the τ-functions $\tau_n \equiv \tau_{F_n}(t, \bar{t}|g)$. Let us begin with the simplest one

$$\tau_1 =< 0_{F_1}|U(t) g \overline{U}(\bar{t})|0_{F_1} >$$

Note that the specific feature of $F = F_1$ is

$$< 0_F|U(t) = \sum_k \mathcal{P}_k(t) < 0_F|T_+^k = \sum_k \mathcal{P}_k(t) < k_F|$$

where the r.h.s. is re-expanded in terms of "generalized Schur polynomials" (the first equality in this formula defines these polynomials) and p states of $F = F_1$ are denoted as $< k_F| =< 0_F|T_+^k$, $k = 0, ..., r = p-1$. Thus,

$$\begin{aligned} \tau_1(t, \bar{t}|g) &= \sum_{k,\bar{k}} \mathcal{P}_k(t)\mathcal{P}_{\bar{k}}(\bar{t}) < 0_F|T_+^k g T_-^{\bar{k}}|0_F >= \\ &= \sum_{k,\bar{k}} \mathcal{P}_k(t)\mathcal{P}_{\bar{k}}(\bar{t}) < k_F|g|\bar{k}_F >= \sum_{k,\bar{k}} \mathcal{P}_k(t) g_{k,\bar{k}} \mathcal{P}_{\bar{k}}(\bar{t}) \end{aligned} \tag{5.25}$$

One can also define

$$\tau_1^{m\bar{m}} \equiv< m_F|U(t) g \overline{U}(\bar{t})|\bar{m}_F >= \sum_{k,\bar{k}} \mathcal{P}_k(t) g_{m+k,\bar{m}+\bar{k}} \mathcal{P}_{\bar{k}}(\bar{t}) \tag{5.26}$$

Now we return to the generic fundamental representation F_n. Since

$$\begin{aligned} &< m_1 \dots m_{n\,F_n}| =< m_{1F}| \otimes < m_{2F}| \otimes \dots \otimes < m_{nF}| + \\ &+ \text{antisymmetrization over } m_1, \dots, m_n = \\ &= \sum_P (-)^P < m_{P(1)}| \otimes < m_{P(2)}| \otimes \dots \otimes < m_{P(n-1)}| \end{aligned} \tag{5.27}$$

the vacuum (highest weight) state F_n can be written as

$$\begin{aligned}
&< 0_{F_n}| =< 0,1,\ldots,n-1_{F_n}| = \\
&= \sum_P (-)^P < P(0)_F|\otimes < P(1)_F|\otimes\ldots\otimes < P(n-1)_F|
\end{aligned} \tag{5.28}$$

Since, for the classical group, (see (5.12))

$$U(t)|_{F_n} = \Delta^{n-1}U(t) = U(t)^{\otimes n}, \quad g|_{F_n} = \Delta^{n-1}(g) = g^{\otimes n} \tag{5.29}$$

one finally gets

$$\begin{aligned}
&\tau_{n+1}(t,\bar t|g) \equiv < 0_{F_n}|U(t)g\overline{U}(\bar t)|0_{F_n} >= \\
&= \sum_{P,\overline{P}} (-)^P(-)^{\overline{P}} \prod_{k=0}^{n} < P(k)_F|U(t)g\overline{U}(\bar t)|\overline{P}(k)_F >= \\
&= \det{}_{0\le m,\overline{m}<n}\tau_1^{m\overline{m}} = \det{}_{0\le m,\overline{m}<n}\sum_{l,\bar l}\mathcal{P}_{l-m}(t)g_{l_F,\bar l_F}\mathcal{P}_{\bar l-\overline{m}}(\bar t) = \\
&= \sum_{\substack{1<m_1<m_2<\ldots\\ 1<\overline{m}_1<\overline{m}_2<\ldots}} \det{}_{ji}\mathcal{P}_{m_j-i}(t)\det{}_{ji}g_{m_j\overline{m}_i}\det{}_{ij}\mathcal{P}_{\overline{m}_i-\bar j}(\bar t)
\end{aligned} \tag{5.30}$$

This result is to be compared with formula (2.24) as $p\to\infty$, with the corresponding Dynkin diagram of the group A_{p-1} being infinite in both directions in this limit (semi-infinite for the forced hierarchy).

The next step is to obtain the differential BI. The determinant formulas (5.30) are, certainly, not the best starting point to this end. It is far more convenient to apply the methods based on the intertwining operators , which we develop in this review. In particular, for the fundamental representations one should consider the fermionic pair $\psi_i^\pm$ $(i=1\ldots p)$ of intertwining operators $\psi^\pm:\ F_1\otimes F_n \rightleftarrows F_{n+1}$. Then, the derivation of BI for arbitrary p almost literally coincides with that for $p\to\infty$, which was described in s.4.1. One only needs to introduce the fermions containing the finite number of degrees of the spectral parameter $\psi^+(z)\equiv\sum_{i=1}^p\psi_i^+z^i$ and $\psi^-(z)\equiv\sum_{i=1}^p\psi_i^-z^{p-i+1}$, and, respectively, to define vertex operators through the formulas

$$\sum_i \Psi_k^{+,i}(t)z^i \equiv \widehat{X}^+(z,t)\tau_n(t), \quad \sum_i \Psi_k^{-,i}(t)z^{p-i+1} \equiv \widehat{X}^-(z,t)\tau_n(t) \tag{5.31}$$

and similarly for $\widehat{\overline{X}}^{\pm}(z,t)$. Then BI acquires the form [28, 29]

$$\oint \frac{dz}{z^{p+2}}\widehat{X}^-(z,t)\tau_n(t,\bar t)\widehat{X}^+(z,t')\tau_m(t',\bar t') = \oint \frac{dz}{z^{p+2}}\widehat{\overline{X}}^-(z,\bar t)\tau_{n+1}(t,\bar t)\widehat{\overline{X}}^+(z,\bar t')\tau_{m-1}(t',\bar t') \tag{5.32}$$

In order to obtain concrete systems of differential equations, one now needs to consider particular choices of parametrizations of $U(t)$, $\overline{U}(\bar t)$.

Conventional parametrization (B). This parametrization leads to the standard KP/Toda hierarchy with determinant representation (2.24)-(2.25) for the τ-function. Indeed, using (5.9), one gets

$$< 0_F|U(t) =< 0_F|\exp\left(\sum_k t_kT_+^{(k)}\right) = \sum_k P_k(t) < k_F|$$

with the conventional Schur polynomials $P_k(t)$ (2.15). The main peculiarity of this evolution is the property

$$\tau_1^{m\overline{m}} = \frac{\partial}{\partial t_m}\frac{\partial}{\partial \bar t_{\overline{m}}}\tau_1 = \left(\frac{\partial}{\partial t_1}\right)^m\left(\frac{\partial}{\partial \bar t_1}\right)^{\overline{m}}\tau_1 \tag{5.33}$$

In order to obtain the system of equations in this parametrization, one can note that the vertex operators (2.33)-(2.34) are

$$\widehat{X}^{+}(z,t) = \mathrm{Pr}_p\left[e^{\xi(z,t)}\mathrm{Pr}\left[z^n e^{-\xi(z^{-1},\widetilde{\partial}_t)}\right]\right]$$
$$\widehat{X}^{-}(z,t) = \mathrm{Pr}_p\left[e^{-\xi(z,t)}\mathrm{Pr}\left[z^{p-n+1} e^{\xi(z^{-1},\widetilde{\partial}_t)}\right]\right] \tag{5.34}$$

(and similarly for the other pair of the vertex operators), where $\mathrm{Pr}[f(z)]$ projects onto the polynomial part of the function $f(z)$ and $\mathrm{Pr}_l[f(z)]$ projects onto the polynomial part of degree l.

Substituting these formulas into (5.32) and expanding this latter into degrees of $t_i - t'_i$, we arrive at the set of equations that gives equations of the standard 2DTL hierarchy as $p \longrightarrow \infty$ [43].

KP/Toda hierarchy in parametrization A. Consider now the same hierarchy with a different evolution given by the parametrization A. From now on, we denote, for brevity, $\widehat{U}(\xi) \equiv U^{(A)}(\xi)$, and the corresponding τ-function is $\widehat{\tau}(\xi,\overline{\xi}|g)$. This τ-function is linear in each time ξ_i and, therefore, has simpler determinant representation and satisfies simpler hierarchy. Indeed, (5.25) now turns into

$$\widehat{\tau}_1(\xi,\overline{\xi}|g) \equiv < 0_{F_1}|\widehat{U}(\xi) g \widehat{\overline{U}}(\overline{\xi})|0_{F_1} >= \sum_{k,\overline{k}\geq 0} s_k \overline{s}_{\overline{k}} < k|g|\overline{k} > \tag{5.35}$$

where $s_k = \xi_1\xi_2\ldots\xi_k$, $s_0 = 1$, and (5.26) is replaced by

$$\widehat{\tau}_1^{m\overline{m}}(\xi,\overline{\xi}|g) \equiv < m_{F_1}|\widehat{U}(\xi) g \widehat{\overline{U}}(\overline{\xi})|\overline{m}_{F_1} >= \frac{1}{s_m\overline{s}_{\overline{m}}}\sum_{\substack{k\geq m\\ \overline{k}\geq\overline{m}}} s_k\overline{s}_{\overline{k}} < k|g|\overline{k} >=$$
$$= \frac{1}{s_m\overline{s}_{\overline{m}}}\sum_{\substack{k\geq m\\ \overline{k}\geq\overline{m}}} \frac{\partial}{\partial \log s_k}\frac{\partial}{\partial\log\overline{s}_{\overline{k}}}\tau_1(\xi,\overline{\xi}|g) = \frac{1}{s_{m-1}\overline{s}_{\overline{m}-1}}\frac{\partial}{\partial\xi_m}\frac{\partial}{\partial\overline{\xi}_{\overline{m}}}\tau_1(\xi,\overline{\xi}|g) \tag{5.36}$$

Therefore,

$$\widehat{\tau}_{n+1} = \det_{0\leq m,\overline{m}\leq n}\widehat{\tau}_1^{m\overline{m}} = \left(\prod_{m=1}^{n} s_m\overline{s}_{\overline{m}}\right)^{-1}\det_{(m,\overline{m})}\left(\sum_{\substack{k\geq m\\ \overline{k}\geq\overline{m}}} s_k\overline{s}_{\overline{k}} < k|g|\overline{k} >\right) =$$
$$= \frac{1}{s_n\overline{s}_n}\sum_{k,\overline{k}\geq n} s_k\overline{s}_{\overline{k}}\det_{0\leq m,\overline{m}\leq n-1}\begin{pmatrix} g_{m\overline{m}} & g_{m\overline{k}}\\ g_{k\overline{m}} & g_{m\overline{m}}\end{pmatrix} \equiv \frac{1}{s_n\overline{s}_n}\sum_{k,\overline{k}\geq n} s_k\overline{s}_{\overline{k}}\mathcal{D}^{(n)}_{k\overline{k}} \tag{5.37}$$

One can compare the determinant representations (2.24)-(2.25) and (5.37) to get the connection between different coordinates t and ξ. This connection possesses the structure $s_k \sim$ some functions of $P_j(t)$. [20]

One can also easily obtain the differential equations for the τ-function in parametrization A.

[20] In the simplest case of the first fundamental representation, one needs to identify $\tau_1(t|g) = \widehat{\tau}_1(\xi|g)$, i.e. $s_k = P_k(t)$, $\frac{\partial}{\partial t_k} = \sum_i s_{i-k}\frac{\partial}{\partial s_i}$. However, identification of $\tau_n(t)$ and $\widehat{\tau}_n(\xi)$ with $n \neq 1$ leads to different relations between ξ and t.

Indeed, it is straightforward to find FBA (2.41) and substitute this into (2.42)

$$\begin{gathered}
\Psi_n^{+,n+k+1}(\xi) = \frac{s_{n+k}}{s_n}\left(\tau_n(\xi) - \xi_n\frac{\partial\tau_n(\xi)}{\partial\xi_n}\right) \\
\Psi_n^{+,n+1}(\xi) = \left(\tau_n(\xi) - \xi_n\frac{\partial\tau_n(\xi)}{\partial\xi_n}\right), \quad \Psi_n^{+,n}(\xi) = -\frac{\partial\tau_n(\xi)}{\partial\xi_n} \\
\Psi_n^{-,k}(\xi) - \xi_{n-1}\frac{\partial\Psi_n^{-,k}(\xi)}{\xi_{n-1}} = \frac{s_{n-1}}{s_{k-1}}\frac{\partial\tau_n(\xi)}{\partial\log\xi_k} + \frac{s_{n-1}}{s_{k-2}}\frac{\partial\tau_n(\xi)}{\partial\xi_{k-1}} \text{ for } k > n \\
\Psi_n^{-,n}(\xi) - \xi_{n-1}\frac{\partial\Psi_n^{-,n}(\xi)}{\xi_{n-1}} = \tau_n(\xi) + \frac{\partial\tau_n(\xi)}{\log\xi_n} \\
\Psi_n^{-,n-1}(\xi) = \xi_{n-1}\tau_n(\xi), \quad \Psi_n^{-,k}(\xi) = 0 \text{ for } k < n-1
\end{gathered} \tag{5.38}$$

As for the values of $\Psi_n^{+,k}(\xi)$ for $k < n$, they are constants that can hardly be expressed as an action of a differential operator on $\tau_n(\xi)$. This means that formula (2.42) (where manifest expressions for $\overline{\Psi}$ analogous to (5.38) can be also easily written down) does not lead to differential equations when k and l are arbitrary choosen. If, however, one chooses $k \le l-1$, because of multiple cancellations due to (5.38), formula(2.42) is almost a differential equation. It can be easily transformed to a differential equation by putting $\xi_{n-1} = 0$ (see (5.38)). One can easily check that the number of independent equations obtained in this way is sufficient to determine the τ-function in full. This means that the whole hierarchy can be still presented in the differential form.

5.4 Quantum case ($q \neq 1$)

q-determinant-like representation. Let us demonstrate now how the technique developed in the previous subsection is deformed to the quantum case, i.e. the group $SL_q(p)$ and, in particular, obtain q-determinant-like representations for the τ-functions similar to (5.30). We also demonstrate that, in parametrization A, formula (5.36) expressing $\tau_1^{m\overline{m}}$ through derivatives of τ_1 is still correct for $q \neq 1$, with all the derivatives replaced by difference operators.

In order to obtain q-determinant-like representations, it suffices to study *any* $U(\xi)$ of the form (5.14), without reference to the particular parametrization A like it was in the case of classical groups.[21]

As a result of absence of the diagonal factor g_D, the classical co-multiplication law (5.29) is replaced in quantum case by the following co-multiplication rule:

$$\Delta^{n-1}(U\{T_i\}) = \prod_{m=1}^{n} U^{(m)} \tag{5.39}$$

where

$$U^{(m)} = U\left\{ I \otimes \ldots\ldots \otimes I \otimes T_i \otimes q^{-2H_i} \otimes \ldots \otimes q^{-2H_i}\right\} \tag{5.40}$$

(T_i appears at the m-th place in the tensor product). Similarly,

$$\overline{U}^{(m)} = \overline{U}\left\{ q^{2H_i} \otimes \ldots\ldots \otimes q^{2H_i} \otimes T_{-i} \otimes I \otimes \ldots \otimes I\right\} \tag{5.41}$$

Let

$$H_i|\bar{j}_{F_1} >= h_{i,\bar{j}}|\bar{j}_{F_1} >, \quad < j_{F_1}|H_i = h_{i,j} < j_{F_1}|$$

[21] Actually, we require the $U(\xi)$ to be an element from $N\mathbf{G}_q$ and to be expressed only through the generators associated with *simple positive* roots: $U(\xi) = U\{\xi_s|T_i\}$. Formula (5.14) is a possible but not unique realization of these requirements.

(in fact, for $SL(p)$ $2h_{i,i-1}=+1,\ 2h_{i,i}=-1$, and all the rest are vanishing). Then

$$\begin{aligned}\tau_n^{j_1\dots j_n\bar{j}_1\dots\bar{j}_n}(\xi_s,\bar{\xi}_s|g) &\equiv \left(\otimes_{m=1}^n < j_m|\right)\Delta^{n-1}(U)\ g^{\otimes n}\ \Delta^{n-1}(\overline{U})\left(\otimes_{m=1}^n|\bar{j}_m>\right) = \\ &= \prod_{m=1}^n\ < j_m|U\left\{T_i q^{-2\sum_{l=m+1}^n h_{i,j_l}}\right\}\ g\ \overline{U}\left\{T_{-i}q^{2\sum_{l=1}^{m-1}h_{i,\bar{j}_l}}\right\}\ |\bar{j}_m>= \\ &= \prod_{m=1}^n \tau_1^{j_m\bar{j}_m}\left(\xi_s q^{-2\sum_{l=m+1}^n h_{i(s),j_l}}, \bar{\xi}_s q^{2\sum_{l=1}^{m-1}h_{i(s),\bar{j}_l}}\right)\end{aligned} \tag{5.42}$$

In order to get a q-determinant-like counterpart of (5.30), one should replace antisymmetrization by q-antisymmetrization in formulas (5.27)-(5.28), since, in the quantum case, the fundamental representations are described by q-antisymmetrized vectors (see details in s.5.2 of [26] and in Appendix 2). We define the q-antisymmetrization as a sum over all permutations

$$([1,\dots,k]_q) = \sum_P (-q)^{\deg\ P}\,(P(1),\dots,P(k)) \tag{5.43}$$

where

$$\deg\ P = \quad \text{number of inversions in } P \tag{5.44}$$

Then, q-antisymmetrizing (5.42) with $j_k = k-1,\ \bar{j}_{\bar{k}} = \bar{k}-1$, one finally gets

$$\tau_n(\xi,\bar{\xi}|g) = \sum_{P,P'}(-q)^{\deg\ P+\deg\ P'}\prod_{m=0}^{n-1}\tau_1^{P(m)P'(\overline{m})}\left(\xi_s q^{-2\sum_{l=m+1}^{n-1}h_{i(s),P(l)}}, \bar{\xi}_s q^{2\sum_{\bar{l}=0}^{m-1}h_{i(s),P'(\bar{l})}}\right) \tag{5.45}$$

If there would be no q-factors twisting the time variables, this would be just a q-determinant which is defined by the formula

$$\det_q A \sim A^{[1}_{[1}\dots A^{n]_q}_{n]_q} = \sum_{P,P'}(-q)^{\deg\ P+\deg\ P'}\prod_a A^{P(a)}_{P'(a)} \tag{5.46}$$

The concrete example of formula (5.45) in the simplest non-trivial case of the second fundamental representation, which makes its structure more transparent is described in Appendix 3.

Note that the q-factors in all the expressions obtained above can be trivially reproduce by action of the operators

$$\begin{aligned} M_j^\pm :&\quad M_j^\pm \xi_s = q^{\pm\delta_{j,i(s)}}\xi_s \\ \overline{M}_j^\pm :&\quad \overline{M}_j^\pm \bar{\xi}_s = q^{\pm\delta_{j,i(s)}}\bar{\xi}_s\end{aligned}$$

We already observed some times that these operators typically emerge in deformations of integrable hierarchies.

Let us now briefly discuss the equations satisfied by the quantum τ-function. For their derivation, one needs to follow the same line as in the classical case and introduce intertwining operators . In quantum case, one should distinguish between the right and left intertwining operators : $\Phi^{\pm,R}:\quad F_n\otimes F_1 \rightleftarrows F_{n+1}$ and $\Phi^{\pm,L}:\quad F_1\otimes F_n \rightleftarrows F_{n+1}$. These operators $\Phi^{\pm,R,L}$ can be expressed through the classical intertwining operators (fermions)[22]:

$$\Phi_i^{\pm,R} = q^{-\sum_{j=1}^{i-1}\psi_j^+\psi_j^-}\psi_i^\pm,\quad \Phi_i^{\pm,L} = q^{\sum_{j=1}^{i-1}\psi_j^+\psi_j^-}\psi_i^\pm \tag{5.47}$$

In analogy with the classical case, one needs to consider now the operator $\Gamma = \sum_i \Phi_i^{+,L}\otimes\Phi_i^{-,R}$ commuting with $g\otimes g$. Then, introducing quantum FBA as averages of the quantum intertwining

[22]This is a corollary of similarity of the irreducible representations of UEA's in the classical and quantum cases (at $|q|\neq 1$) [71].

operators $\Phi^\pm$ (properly labeled by indices L and R) and corresponding vertex operators, one obtains the same equations (2.39)-(5.32) but with re-defined entries. Technically, the action of vertex operators can be calculated with the help of formula (5.47) and leads to quantum counterparts of formulas (2.31)-(2.32).

Now we fix the concrete parametrization A and show how all these formulas work.

Parametrization A. Most of expressions of the previous subsection remain almost the same in this parametrization (although, in the quantum case, one needs to take care of normal ordering of different objects). In particular,

$$\widehat{\tau}_1(\xi,\overline{\xi}|g) \equiv < 0_{F_1}|\widehat{U}(\xi)g\widehat{\overline{U}}(\overline{\xi})|0_{F_1} > = \sum_{k,\overline{k}\geq 0} s_k\overline{s}_{\overline{k}} < k|g|\overline{k} > \tag{5.48}$$

where again $s_k = \xi_1\xi_2\ldots\xi_k$, $s_0 = 1$, while $\overline{s}_k = \overline{\xi}_k\ldots\overline{\xi}_2\overline{\xi}_1$, $\overline{s}_0 = 1$ and

$$\widehat{\tau}_1^{m\overline{m}}(\xi,\overline{\xi}|g) = s_m^{-1}\left(\sum_{\substack{k\geq m\\ \overline{k}\geq\overline{m}}} s_k\overline{s}_{\overline{k}} < k|g|\overline{k} >\right)\overline{s}_{\overline{m}}^{-1} = s_{m-1}^{-1}\left(D_{\xi_m}\overline{D}_{\overline{\xi}_{\overline{m}}}\tau_1(\xi,\overline{\xi}|g)\right)\overline{s}_{\overline{m}-1}^{-1} \tag{5.49}$$

Here[23] $D_{\xi_i}f(\xi) \equiv \frac{1}{\xi_i}\frac{M_i^{+2}-1}{q^2-1}f(\xi)$, $\overline{D}_{\overline{\xi}_i}f(\overline{\xi}) \equiv \left[\frac{M_i^{-2}-1}{q^{-2}-1}f(\overline{\xi})\right]\frac{1}{\overline{\xi}_i}$ (in these operators, the order is also crucial!). Now one can manifestly express τ_n through τ_1, using formulas (5.45) and (5.49). In Appendix 3, we describe how it works for the simplest case of the second fundamental representation.

FBA for the τ-function in parametrization A is given by the following expressions

$$\begin{gathered}
\Psi_n^{+,n+k+1}(\xi) = q^{n+1}s_n^{-1}s_{n+k}M_{n+1}^+\ldots M_{n+k-1}^+\left(\tau_n(\xi)-\xi_nD_n\tau_n(\xi)\right)\\
\Psi_n^{+,n+1}(\xi) = q^{n+1}\left(\tau_n(\xi)-\xi_nD_n\tau_n(\xi)\right),\quad \Psi_n^{+,n}(\xi) = -q^nD_n\tau_n(\xi)\\
\Psi_n^{-,k}(\xi)-\xi_{n-1}D_{n-1}\Psi_n^{-,k}(\xi) =\\
= q^{n-2}s_{k-2}^{-1}s_{n-1}D_{i-1}\tau_n(\xi)+q^{n-2}s_{k-1}^{-1}s_{n-1}\xi_iD_i\tau_n(\xi) \text{ for } k>n\\
\Psi_n^{-,n}(\xi)-\xi_{n-1}D_{n-1}\Psi_n^{-,n}(\xi) = q^{n-2}\tau_n(\xi)+\xi_nD_n\tau_n(\xi)\\
\Psi_n^{-,n-1}(\xi) = q^{n-2}\xi_{n-1}\tau_n(\xi),\quad \Psi_n^{-,k}(\xi) = 0 \text{ for } k<n-1
\end{gathered} \tag{5.50}$$

Substituting these expressions in (2.42), one obtains a set of equations which is a quantum counterpart of the KP/Toda hierarchy in parametrization A.

To conclude this section, point out once more that the determinant representations of the classical τ-functions do not turn into exact q-determinant representations at $q\neq 1$. The reason for this is that, in the quantum case, the evolution operator is no longer the group element. This happens because no nilpotent subgroup $N\mathbf{G}_q$ exists in the quantum group. To avoid this problem, one could begin from a slightly different parametrization of the τ-function such that the evolution operator lies in the Borel (not just nilpotent) subgroup $B\mathbf{G}_q$. In the classical limit, additional Cartan generators can be removed by redefinition of the element g (labelling the point of the Grassmannian). However, in the quantum case the Cartan part of the evolution would essentially change the result: the evolution operator becomes a group element (for $B\mathbf{G}_q$) and all "twists" of times in formula (5.45) disappear. Thus defined τ-function is just the q-determinant. In order to fulfil the whole scheme, one still needs to find an appropriate parametrization of (a set of) group elements $B\mathbf{G}_q$ by exactly $r_\mathbf{G}$ "times".

[23] There is an ambiguity in the choice of these operators as the τ-function is a linear function of times and, therefore, any linear operator which makes unity from ξ is suitable. We fix them to act naturally on the q-exponential and correspond to the difference operators that we used throughout this review.

6 Wave function and S-matrix in quantum Liouville (Toda) theory

As it has been demonstrated above, the group acting in the space of the spectral parameter, i.e. on solutions of integrable system, in the course of quantization should be merely replaced by the corresponding quantum group. In essence, it is the very sense of the quantization procedure. Now we are going to consider an example of some different group structure that can be also manifested in the Toda theory. It is the group acting in "the space-time", i.e. on variables of equation. It is the remarkable property of this group that it survives in the course of quantization, but the way of dealing with it as well as the objects under consideration do completely change.

Namely, consider again the Toda molecule additionally reduced to the Toda chain, i.e. depending only on the sum of times. Let us demonstrate that this system can be obtained via the Hamiltonian reduction of a free system given on the cotangent bundle for a Lie group [32, 72].

6.1 Classical Toda system as Hamiltonian reduction

Indeed, consider the cotangent bundle $T^*\mathbf{G}$ for the real split group G. By the group shifts, it can be reduced to the pair (Y, g), $Y \in \mathcal{G}^*$, $g \in \mathbf{G}$, where $\mathcal{G}$ is the algebra corresponding to the group $\mathbf{G}$. There is the canonical bi-invariant symplectic form on $T^*\mathbf{G}$

$$\omega = \delta Y(\delta g g^{-1}) \tag{6.1}$$

and the set of invariant commuting Hamiltonians

$$\frac{< Y^{d_k} >}{d_k}, k = 1\ldots, r \tag{6.2}$$

where $d_k = 2, \ldots$ are invariants of $\mathcal{G}$ and Y^{d_k} are $ad^*\mathbf{G}$-invariant polynomials on $\mathcal{G}^*$. It is the upstairs Hamiltonian system.

Now consider the symplectic reduction on $T^*\mathbf{G}$ (i.e. the reduction preserving the symplectic form ω (6.1)) w.r.t. the action of the left and right nilpotent subgroups $\overline{N} \oplus N$ of the group G

$$\begin{gathered} g \mapsto vg, \ Y \mapsto vYv^{-1}, \quad v \in \overline{N}, \\ g \mapsto gn, \ Y \mapsto Y, \quad n \in N \end{gathered} \tag{6.3}$$

It gives the two moment maps

$$\mu_v = Pr_{\overline{\mathcal{N}}^*}\, Y, \quad \mu_n = Pr_{\mathcal{N}^*}\, g^{-1}Yg \tag{6.4}$$

Making use of the Gauss decomposition, one can transform g by (6.3) to the Cartan subgroup A. Let $g = \exp\phi$, $\phi \in \mathcal{A}$. Assume that

$$\mu_v = Pr_{\overline{\mathcal{N}}^*}\, Y = \mu^L, \quad \mu_n = Pr_{\mathcal{N}^*}\, g^{-1}Yg = \mu^R \tag{6.5}$$

where $\mu^L = \sum_{\alpha\in\Pi} \mu^L_\alpha \mathcal{G}_\alpha$, $\mu^R = \sum_{\alpha\in\Pi} \mu^R_\alpha \mathcal{G}_{-\alpha}$, $\mu^{R,L}_\alpha$ are arbitrary constants and Π denotes the set of all positive roots. After "diagonalizing" g by the Gauss decomposition at the point $g = \exp\phi$, $\phi \in \mathcal{A} = h_{\mathbf{G}}$, one can manifestly solve the constraint (6.5)

$$Y = p + \sum_{\alpha\in\Pi} \left(\mu^R_\alpha e^{\alpha(\phi)}\mathcal{G}_\alpha + \mu^L_\alpha \mathcal{G}_{-\alpha}\right) \tag{6.6}$$

Then, the reduced symplectic form acquires the canonical form

$$\omega^{red} = \delta p \delta \phi = \sum_{k=1}^{r} \delta p_k \delta \phi_k \tag{6.7}$$

and the reduced phase space $\overline{N} \setminus \backslash T^*\mathbf{G}//N$ is of dimension

$$2 \dim \mathbf{G} - 2 \dim \overline{N} - 2 \dim N = 2r \tag{6.8}$$

Combination $W(Y; \tau_1 \ldots, \tau_r) = \sum_{k=1}^{r} \frac{\tau_k}{d_k} < Y^{d_k} >$ defines the commuting Hamiltonians of the classical hierarchy given by the Lax operator (6.6). In particular,

$$< Y^2 >= \frac{1}{2} \sum_{k=1}^{r} p_k^2 + \sum_{\alpha \in \Pi} \mu_\alpha e^{2\alpha(\phi)} \tag{6.9}$$

is the conventional Toda chain Hamiltonian [73]. The classical action in this representation takes the form

$$S^I = \int Y(\partial_t g g^{-1}) - < Y^2 > + < B_L, Y - \mu^L > + < B_R, g^{-1} Y g - \mu^R > \tag{6.10}$$

where $B_L \in \overline{\mathcal{N}}, \ B_R \in \mathcal{N}$ are the Lagrange multipliers.

In order to compare with the standard Toda chain formulas, it suffices to choose as $\mathbf{G}$ the group $SL(p)$ and consider its fundamental representation. Then, Y (6.6) coincides with the Toda chain Lax operator etc.

The above construction is essentially based on the Gauss decomposition [30]. In fact, the same system can be obtained via Iwasawa decomposition. Just this latter decomposition is the standard one used so far in studies of these system. In particular, in the next subsections we consider quantization of the Toda system, which has been originally performed in the Iwasawa decomposition framework and led to the so-called Whittaker wave function (WF). However, we use everywhere the Gauss decomposition that is more applicable to the affine algebras [30]. One can find in [30] further details and some discussion of the connection of the two approaches – the Gauss and Iwasawa ones.

6.2 Solution to the Liouville quantum mechanics – general scheme

A non-trivial property of the "space-time" group manifested above[24] in the $SL(p)$ Toda molecule is that, in the course of quantization, this group remains classical, while its interpretation changes: quantization implies an orbit interpretation. Indeed, while the classical system is obtained via the Hamiltonian reduction of a free system given on the cotangent bundle for a simple real Lie group [32], the quantum model is related to irreducible unitary representations of *the same* group. Thus, the quantum system should be rather interpreted within the geometrical quantization approach [33].

Let us now briefly describe the general approach that allows one to solve differential equations for the eigenvalues (EV) and, in particular, the Schroedinger equation, in group theory terms. After this, we consider concrete examples. Technically one needs to make some steps.

1) Let $g(\xi|T) \in A_{\mathbf{G}}(\xi) \otimes U_{\mathcal{G}}$ be "the universal group element" (see s.4.3) of the Lie group $\mathbf{G}$,[25] where ξ somehow parametrizes the group manifold and T are generators of $\mathbf{G}$ in some (not obligatory irreducible) representation, their only property being $[T^a, T^b] = f^{abc}T^c$.

[24] This space-time group is not to be mixed with the group $SL(p)$ describing the reduction to the Toda molecule (see s.3). In order to obtain that these two are absolutely different, one can remark that the two-dimensional Toda molecule is described by the same reduction group $SL(p)$ but by the affine space-time group $\widehat{SL}(p)$.

[25] This can also be a quantum group, which leads, however, to the difference Liouville equation – see [74].

2) For any given parametrization $\{\xi\}$, one can introduce two sets of differential operators $\mathcal{D}_{R,L}(\xi)$ such that

$$\begin{aligned} \mathcal{D}^a_L(\xi) g(\xi|T) &= T^a g(\xi|T) \\ \mathcal{D}^a_R(\xi) g(\xi|T) &= g(\xi|T) T^a \end{aligned} \tag{6.11}$$

These operators satisfy the obvious commutation relations

$$\begin{aligned} \left[\mathcal{D}^a_L, \mathcal{D}^b_L\right] &= -f^{abc}\mathcal{D}^c_L \\ \left[\mathcal{D}^a_R, \mathcal{D}^b_R\right] &= f^{abc}\mathcal{D}^c_R \\ \left[\mathcal{D}^a_L, \mathcal{D}^b_R\right] &= 0 \end{aligned} \tag{6.12}$$

3) For a given representation $\mathcal{R}$, scalar product $< \ | \ >$ and two elements of the representation $< \psi_L|$ and $|\psi_R >$ construct the matrix element

$$F_{\mathcal{R}}(\xi|\psi_L, \psi_R) =< \psi_L|g(\xi|T)|\psi_R > \tag{6.13}$$

The action of any combination of the differential operators $\mathcal{D}_R$ on F inserts the same combination of generators T to the right of $g(\xi|T)$. If $|\psi_R >$ happens to be an eigenvector of this combination of generators, F provides a solution to the corresponding differential equation (as the eigenvalue problem).

4) Of special interest are the Casimir operators, since $|\psi_R >$ which are their eigenvectors are just elements of irreducible representations.

5) In particular, the quadratic Casimir operator, when expressed through $\mathcal{D}_R$, is the Laplace operator and gives rise to some important Schroedinger equations, of which the equation for the Toda system is a typical example.

In fact, in order to get the Schroedinger equation for the Toda-type system, one needs to impose additional constraints on the states $< \psi_L|$ and $|\psi_R >$, which correspond to the Hamiltonian reduction (6.3) of the free motion on the group manifold of $\mathbf{G}$ (in the classical case). Such a reduction, as we saw in the previous subsection, is usually associated with some decomposition of $\mathbf{G}$ into a product of subgroups, and different decompositions can lead to equivalent reductions. Essentially, reduction allows one to get rid of the dependence of $F(\xi)$ on some of the coordinates ξ. In the case of finite dimensional Lie groups, the remaining coordinates can be choosen to correspond to the Cartan torus, while, for affine algebras, it is more natural to preserve the dependence on all the diagonal matrices. The matrix element we discuss here, WF F, is sometimes called Whittaker function.

Let us remark that the differential operators $\mathcal{D}_L$ and $\mathcal{D}_R$ realize respectively the left and the right regular representations of the algebra of $\mathbf{G}$ (in fact, the left one is antirepresentation), which can be invariantly given by the action on the algebra $A_{\mathbf{G}}$ of functions on the group:

$$\pi^L_{reg}(h)f(g) = f(hg), \quad \pi^R_{reg}(h)f(g) = f(gh), \quad g, h \in G \tag{6.14}$$

Manifestly these operators can be constructed in the following way. Let us consider the group element g and the (formal) differential operator d acting as the full derivative on functions of ξ_i, i.e. $\mathrm{d} \equiv \sum_i d\xi_i \frac{\partial}{\partial \xi_i}$. Then, one may calculate $g^{-1}\cdot \mathrm{d}g$ (Maurer-Cartan form) and expand it into the generators of the algebra:

$$g^{-1}\cdot \mathrm{d}g = \sum_{a,i} c_{a,i} T_a d\xi_i \tag{6.15}$$

i.e.

$$\mathrm{d}g = \sum_{a,i} c_{a,i} \left(\mathcal{D}^a_R g\right) d\xi_i \tag{6.16}$$

Now one reads off the manifest form of the differential operators $\mathcal{D}_R$ from this expression. Analogously, one can calculate $\mathcal{D}_L$.

Note that it suffices to consider g only in the fundamental representation, since matrix elements in this representation generate the whole algebra $A_{\mathbf{G}}$ and the group action can be extended onto the whole algebra using the co-multiplication. At the same time, calculating the coefficients $c_{a,i}$ in the fundamental representation is quite easy.

As an instance of the general scheme, we briefly consider below the example of the Schroedinger equation for the Toda-type system, while the details can be found in [30]. We begin with the simplest case of the group $SL(2)$, i.e. the Liouville quantum mechanics. The problem we address to is to find an integral representation for the WF and its asymptotics that define the scattering S-matrix.

6.3 Liouville quantum mechanics – $SL(2,\mathbf{R})$

Liouville system for the group $SL(2,\mathbf{R})$. The corresponding Lie algebra is given by the relations

$$[T_+,T_-]=T_0,\quad [T_\pm,T_0]=\mp 2T_\pm \tag{6.17}$$

In the fundamental representation

$$T_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ T_+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ T_-=\begin{pmatrix}0&0\\1&0\end{pmatrix} \tag{6.18}$$

The quadratic Casimir operator is equal

$$C=(T_-T_++T_+T_-)+\frac{1}{2}T_0^2=2T_-T_++T_0+\frac{1}{2}T_0^2 \tag{6.19}$$

Let us now perform the first step of our general scheme and parametrize the group element as follows[26]

$$g(\psi,\phi,\chi|T)=e^{\psi T_-}e^{\phi T_0}e^{\chi T_+} \tag{6.20}$$

In the fundamental representation, the group element has the form

$$g=\begin{pmatrix}1&0\\\psi&1\end{pmatrix}\begin{pmatrix}e^\phi&0\\0&e^{-\phi}\end{pmatrix}\begin{pmatrix}1&\chi\\0&1\end{pmatrix}=\begin{pmatrix}e^\phi&\chi e^\phi\\\psi e^\phi&\psi\chi e^\phi+e^{-\phi}\end{pmatrix} \tag{6.21}$$

As the next step, we need to construct differential operators realizing the right and the left regular representations. We already remarked that, to this end, one should calculate "the currents" $g^{-1}\cdot \mathrm{d}g$ and $\mathrm{d}g\cdot g^{-1}$:

$$\begin{aligned}g^{-1}dg&=\begin{pmatrix}-e^{2\phi}\chi d\psi+d\phi & -e^{2\phi}\chi^2 d\psi+2\chi d\phi+d\chi\\ e^{2\phi}d\psi & e^{2\phi}\chi d\psi-d\phi\end{pmatrix}\\ dg\cdot g^{-1}&=\begin{pmatrix}-e^{2\phi}\psi d\chi+d\phi & e^{2\phi}d\chi\\ -e^{2\phi}\psi^2 d\chi+2\psi d\phi+d\psi & e^{2\phi}\psi d\chi-d\phi\end{pmatrix}\end{aligned} \tag{6.22}$$

[26]This parametrization differs from that used in the previous sections by the permutation of the nilpotent subgroups, which is given by action of the inner group automorphism – antipode.

Using this formula and (6.15), one easily gets

$$\begin{aligned}\frac{\partial g}{\partial\phi} &= g(T^0+2\chi T^+) = (T^0+2\psi T^-)g \\ \frac{\partial g}{\partial\chi} &= gT^+ = e^{2\phi}(T^+-\psi T^0-\psi^2T^-)g \\ \frac{\partial g}{\partial\psi} &= g(-\chi^2T^+-\chi T^0+T^-)e^{2\phi} = T^-g\end{aligned} \tag{6.23}$$

and, using (6.16), –

$$\mathcal{D}_R^+ = \frac{\partial}{\partial\chi},\ \mathcal{D}_R^0 = -2\chi\frac{\partial}{\partial\chi}+\frac{\partial}{\partial\phi},\ \mathcal{D}_R^- = e^{-2\phi}\frac{\partial}{\partial\psi}+\chi\frac{\partial}{\partial\phi}-\chi^2\frac{\partial}{\partial\chi} \tag{6.24}$$

$$\mathcal{D}_L^- = \frac{\partial}{\partial\psi},\ \mathcal{D}_L^0 = -2\psi\frac{\partial}{\partial\psi}+\frac{\partial}{\partial\phi},\ \mathcal{D}_L^+ = -\psi^2\frac{\partial}{\partial\psi}+\psi\frac{\partial}{\partial\phi}+e^{-2\phi}\frac{\partial}{\partial\chi} \tag{6.25}$$

At the third step, we need to choose some representation. Consider the principal (spherical) series of representations induced by one-dimensional representations of the Borel subalgebra [75]. The space of the representation consists of functions of one variable and matrix elements (scalar product) are given by integrals with the flat measure. Action of the algebra is given by the differential operators

$$T_+ = \frac{\partial}{\partial x},\ T_0 = -2x\frac{\partial}{\partial x}+2j,\ T_- = -x^2\frac{\partial}{\partial x}+2jx \tag{6.26}$$

where j is spin of the representation. We consider here only unitary representations, since, for these representations only, the scalar product is given by a convergent integral. The unitarity condition implies that $j+\frac{1}{2}$ is pure imaginary[27].

The fourth step implies that we consider the quadratic Casimir operator (it can be calculated both in the left and in the right regular representations):

$$C = \frac{1}{2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial}{\partial\phi}+2e^{-2\phi}\frac{\partial^2}{\partial\psi\partial\chi} \tag{6.27}$$

Its EV in the spin j representation is $2(j^2+j)$. Therefore, the matrix element $F = <\psi_L|g(\theta,\phi,\chi)|\psi_R>$ with $<\psi_L|$ and $\psi_R|$ belonging to the spin j representation satisfies the equation

$$CF = 2(j^2+j)F \tag{6.28}$$

As the last, fifth step we need to fix the reduction conditions. We choose them to be

$$\frac{\partial}{\partial\chi}F_G = i\mu_R F_G,\ \frac{\partial}{\partial\psi}F_G = i\mu_L F_G \tag{6.29}$$

i.e. (see (6.24) and (6.25))

$$T_+|\psi_R> = i\mu_R|\psi_R>,\quad <\psi_L|T_- = i\mu_L<\psi_L| \tag{6.30}$$

Under this reduction, the Hamiltonian (quadratic Casimir operator) is equal to (cf. (6.9))

$$H = \frac{1}{2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial}{\partial\phi}-2\mu_R\mu_L e^{-2\phi} \tag{6.31}$$

[27] Similarly, for the general $SL(p)$, all $j_i+\frac{1}{2}$ (i.e. $\boldsymbol{j}+\boldsymbol{\rho}$) should be pure imaginary.

and the function $\Psi(\phi) = e^{\phi}F$ satisfies the following Schroedinger equation

$$\left[\frac{1}{2}\frac{\partial^2}{\partial\phi^2} - 2\mu_R\mu_L e^{-2\phi}\right]\Psi(\phi) = 2\left(j+\frac{1}{2}\right)^2\Psi(\phi) \equiv \lambda^2\Psi(\phi) \tag{6.32}$$

which is actually the Liouville Schroedinger equation.

Note that, using the representation (6.21) and the conditions (6.30), one can immediately obtain the equation (6.31) through the following chain of formulas[28]:

$$\begin{gathered}2j(j+1)F_G^{(j)} \equiv 2j(j+1) < \psi_L|e^{\phi T_0}|\psi_R >_j = < \psi_L|e^{\psi T_-}e^{\phi T_0}e^{\chi T_+}\widehat{C}|\psi_R >_j = \\ = < \psi_L|e^{\phi T_0}(2T_-T_+ + T_0 + \frac{1}{2}T_0^2)|\psi_R >_j = \left(-2\mu_R\mu_L e^{-2\phi} + \frac{\partial}{\partial\phi} + \frac{1}{2}\frac{\partial^2}{\partial\phi^2}\right) < \psi_L|e^{\phi T_0}|\psi_R >_j\end{gathered} \tag{6.33}$$

Solving the Liouville Schroedinger equation. Now we should only solve the reduction conditions (6.30) and find out some explicit integral representation for the WF. We use formulas (6.26). Then, the conditions (6.30) take the form

$$\begin{gathered}T_+|\psi_R> = \frac{\partial}{\partial x}\psi_R(x) = i\mu_R\psi_R(x), \\ < \psi_L|T_- = (2x + x^2\frac{\partial}{\partial x} + 2jx)\psi_L(x) = i\mu_L\psi_L(x)\end{gathered} \tag{6.34}$$

Their solutions are

$$\psi_R(x) = e^{i\mu_R x}, \quad \psi_L(x) = x^{-2(j+1)}e^{-\frac{i\mu_L}{x}} \tag{6.35}$$

This finally provides us with the solution to the Schroedinger equation (6.32):

$$\begin{aligned}e^{\phi}F^{(j)}(\phi) &= e^{\phi} < \psi_L|e^{\phi T_0}|\psi_R >_j = e^{\phi}\int x^{-2(j+1)}e^{-\frac{i\mu_L}{x}}e^{\phi(2j-2x\frac{\partial}{\partial x})}e^{i\mu_R x}\, dx = \\ &= \left(\frac{i}{\mu_R}\right)^{-(2j+1)} e^{-(2j+1)\phi}\int_0^\infty x^{-2(j+1)}e^{-\frac{\mu_L\mu_R e^{-2\phi}}{x} - x}dx = \\ &= 2\left(i\sqrt{\frac{\mu_L}{\mu_R}}\right)^{-(2j+1)} K_{2j+1}(2\sqrt{\mu_L\mu_R}e^{-\phi})\, dx\end{aligned} \tag{6.36}$$

where $K_\nu(z)$ is the Macdonald function [76].

Harish-Chandra function and its asymptotics. Let us once more return to the Schroedinger equation (6.32). The condition of unitarity of some representation implies pure imaginary $\lambda \equiv \sqrt{2}(j+\frac{1}{2})$, i.e. continuous spectrum of the Schroedinger equation (and real energy). Solutions of this equation obviously oscillate at only one infinity, while exponentially decreasing at the other one. We can consider the scattering problem. Then, the reflection S-matrix is equal to the ratio of the coefficients of falling and reflected waves. We really need to find the exponential asymptotics $e^{\pm\lambda\phi}$ at *the same* infinity. Technically, it can be done as follows: one considers a small real shift of λ so that, depending on its sign, one can extract one of the two asymptotics.

We perform this procedure for the Macdonald function:

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-\frac{z^2}{4t}}t^{-\nu-1}dt \tag{6.37}$$

[28] From now on, we consider only the matrix element of $e^{\phi_I T_0}$, instead of g, in order to exclude trivial ψ- and χ-dependencies of F.

If $\mathrm{Re}\nu \le 0$, one can through away the asymptotically (i.e. at small z) small term $\frac{z^2}{4t}$ so that the integral is equal to

$$K_\nu(z) \underset{z\to 0}{\sim} \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t} t^{-\nu-1} dt = -\frac{\pi}{2} \frac{1}{\sin \pi\nu \Gamma(1+\nu)} \left(\frac{z}{2}\right)^\nu \tag{6.38}$$

On the other hand, at $\mathrm{Re}\nu \ge 0$ one needs to make the replace $t \to z^2 t$ and the result reads as

$$K_\nu(z) \underset{z\to 0}{\sim} \frac{1}{2}(2z)^{-\nu} \int_0^\infty e^{-\frac{1}{4t}} t^{-\nu-1} dt = \frac{\pi}{2} \frac{1}{\sin \pi\nu \Gamma(1-\nu)} \left(\frac{z}{2}\right)^{-\nu} \tag{6.39}$$

Note that, though the ratio of the two calculated asymptotics is unambiguously defined, each of them separately is defined up to a common normalization factor of the WF. This factor is used to fix partially requiring the absence of poles at finite values of momentum. The issue of the entirely fixed normalization is discussed in [30].

In this concrete case, the condition of absence of the poles implies that the WF (6.36) is to be multiplied by the function $\sin \pi\lambda$ that, cancelling all the poles, introduce new additional zeroes. The two asymptotics of the so-normalized WF are[29]

$$c_\pm = \frac{1}{\Gamma(1 \pm \lambda)} \tag{6.40}$$

These functions are called Harish-Chandra functions [77] and play an important role in the group theory.

6.4 Toda quantum mechanics

Quantum mechanics for the Toda molecule – $SL(p, \mathbf{R})$**.** We apply now the above developed general scheme to more general case of the group $SL(p, R)$. Calculations in this case are quite similar to the Liouville case, however, they require some additional information on the structure of the unitary representations of $SL(p, R)$. Therefore, we write down here only the results and refer to Appendix 4 and, especially, the reference [30] which contain the detailed calculations.

Namely, like the previous subsection, we consider the matrix element F given in some unitary representation of the group and act on it by some Casimir operator. Since the group $SL(p, R)$ is of rank $p-1$, i.e. has $p-1$ independent Casimir operators, the matrix element F satisfies now $p-1$ differential equations for the EV simultaneously. Thus, this matrix element solves the quantum problem with the Hamiltonians of s.6.1 corresponding to the Toda system. Among these Hamiltonians, there is the quadratic Hamiltonian (6.9) that gives rises to the second order equation, that is to say Schroedinger equation. It is of the form

$$\left(\frac{\partial^2}{\partial \boldsymbol{\phi}^2} - 2\sum_i \mu_i^L \mu_i^R e^{\boldsymbol{\alpha}_i \boldsymbol{\phi}}\right) \Psi^{(\boldsymbol{\lambda})}(\boldsymbol{\phi}) = \boldsymbol{\lambda}^2 \Psi^{(\boldsymbol{\lambda})}(\boldsymbol{\phi}) \tag{6.41}$$

the function $\Psi^{(\boldsymbol{\lambda})}(\boldsymbol{\phi})$ being the rescaled matrix element $e^{-\boldsymbol{\rho\phi}} F^{(\boldsymbol{\lambda})}(\boldsymbol{\phi})$ and $\mu_i^{R,L}$ being a set of the cosmological constants. Manifestly calculating this matrix element, one gets the integral solution to this equation:

$$\Psi(\phi) = e^{-\boldsymbol{\lambda\phi}} \int \prod_{i<j} dx_{ij} \prod_{i=1}^{p-1} \Delta_i^{-(\boldsymbol{\lambda\alpha}_i+1)}(xS^{-1}) \times e^{i\mu_i^R x_{i,i+1} e^{\boldsymbol{\alpha}_i \boldsymbol{\phi}} - i\mu_{p-i}^L \frac{\Delta_{i,i+1}(xS^{-1})}{\Delta_i(xS^{-1})}} \tag{6.42}$$

[29]Hereafter, we omit the trivial factor depending on the cosmological constant $\mu_L \mu_R$.

Here $\boldsymbol{\alpha}_i$ are the positive simple roots, S is the inner automorphism of the group $SL(p)$, which maps the upper-triangle matrices to the lower-triangle ones, $S_{ij} \equiv \delta_{i+j,p+1}$, $\Delta_i(A)$ is the upper main $i \times i$ minor of the matrix A and $\Delta_{i,i+1}(A)$ is the same minor of the matrix with i and $i+1$ columns exchanged.

Now looking at the asymptotics of this WF, one can obtain the Harish-Chandra functions [30]. First of all, note that the number of different asymptotics coincides with the number of elements of the Weyl group for $SL(p)$, while the WF (6.42) is expressed through only the simple roots. Thus, it is natural that the Harish-Chandra functions are labeled by elements of the Weyl group and are related by action of this group. In order to calculate these asymptotics, one may use the standard technique developed for the Iwasawa decomposition case [35] and the result reads as

$$c_s(\boldsymbol{\lambda}) = \prod_{\boldsymbol{\alpha}\in\Delta^+} \frac{1}{\Gamma(1 + s\boldsymbol{\lambda}\cdot\boldsymbol{\alpha})} \tag{6.43}$$

where s denotes elements of the Weyl group and the product runs over all the positive roots. The ratio of these Harish-Chandra functions gives us the reflection S-matrix of the theory.

Quantum field Liouville system. As the next step, we extend our construction to the affine case. Indeed, considering the group $\widehat{SL(2)}$ and the "point-wise" Gauss decomposition (used for bosonization of affine algebras – see [37]), one obtains the Schroedinger equation for the two-dimensional *field* Liouville system. This means that, calculating the Harish-Chandra functions, we get in this case the S-matrix (or the two-point correlation function) for the two-dimensional Liouville field theory. Indeed, choose the system of positive roots in this case as follows

$$\boldsymbol{\alpha}_0 + n(\boldsymbol{\alpha}_0+\boldsymbol{\alpha}_1),\ n(\boldsymbol{\alpha}_0+\boldsymbol{\alpha}_1),\ \boldsymbol{\alpha}_1 + n(\boldsymbol{\alpha}_0+\boldsymbol{\alpha}_1),\ n = 0,1,2,\dots \tag{6.44}$$

Denote

$$\boldsymbol{\lambda}\cdot\boldsymbol{\alpha}_0 = \frac{1}{2} - p + \tau, \quad \boldsymbol{\lambda}\cdot\boldsymbol{\alpha}_1 = -\frac{1}{2} + p \tag{6.45}$$

Then, one can use formula (6.43) for the Harish-Chandra functions only shifting the argument of the Γ-function by $1/2$ because of the affine situation:

$$c(\boldsymbol{\lambda}) = \prod_{n\geq 0}\Gamma^{-1}(p+n\tau)\prod_{n\geq 1}\Gamma^{-1}(n\tau)\Gamma^{-1}(1-p+n\tau) \tag{6.46}$$

This expression certainly requires a careful regularization, but all the infinite products cancel from the corresponding reflection S-matrix (two-point function)

$$S(p) = \frac{c(-p)}{c(p)} = \frac{\Gamma(1+p)\Gamma(1+\frac{p}{\tau})}{\Gamma(1-p)\Gamma(1-\frac{p}{\tau})} \tag{6.47}$$

This expression coincides with the two-point functions obtained in the papers [39, 38] in a very different way[30].

Now we introduce a fundamental building block – the function

$$i(\boldsymbol{\lambda}) = i(p,\tau) \sim \prod_{m,n\geq 0}(p+m+n\tau) \tag{6.48}$$

where the product goes over only one (positive) quadrant in the plane m,n. One can construct from this block both the Harish-Chandra function (6.46) and the elliptic theta-functions

$$\theta(p+\frac{1}{2}+\frac{\tau}{2},\tau) \sim i(p,\tau)i(-p,\tau)i(p,-\tau)i(-p,-\tau) \tag{6.49}$$

[30]In notations of [38], $p = 2iP/b$ and $\tau = b^2$.

and also the q-exponential

$$e_q(e^{2\pi ip}) \sim \frac{1}{i(p,\tau)i(-p,-\tau)}, \quad 1 = e^{i\pi\tau} \tag{6.50}$$

just by taking products over different quadrants. All these expressions certainly require some accurate regularization, since the unrestricted products diverge. The regularized expressions can be found in [78] and [38].

Appendix 1 Forced hierarchies

In this Appendix, we reproduce some technical calculations omitted from s.2.3 [52, 46]. Namely, we consider the forced hierarchy given by the condition

$$\tau_n = 0 \ , \ n < 0 \tag{A1.1}$$

This corresponds to choosing the element of the Grassmannian in the form

$$G = G_0 P_+ \tag{A1.2}$$

where P_+ is the projector onto the positive states:

$$P_+|n\rangle = \theta(n)|n\rangle \tag{A1.3}$$

which is realized as the fermionic operator

$$P_+ =: \exp\left[\sum_{i<0} \psi_i \psi_i^*\right] : \tag{A1.4}$$

and enjoys the properties

$$P_+\psi^*_{-k} = \psi_{-k}P_+ = 0 \ , \ k > 0 \tag{A1.5}$$

$$[P_+, \psi_k] = [P_+, \psi_k^*] = 0 \ , \ k \geq 0 \tag{A1.6}$$

$$P_+^2 = P_+ \tag{A1.7}$$

We also fix G_0 to include only positive fermionic modes ψ_k and ψ_k^* at $k \geq 0$:

$$G_0 =: \exp\left\{\left(\int_\gamma A(z,w)\psi_+(z)\psi_+^*(w^{-1})dzdw\right) - \sum_{i\geq 0}\psi_i\psi_i^*\right\} : \tag{A1.8}$$

where $\psi_+(z) = \sum_{k\geq 0}\psi_k z^k$, $\psi_+^*(z) = \sum_{k\geq 0}\psi_k^* z^{-k}$ and γ is some integration domain. Besides, we need the projector onto the negative states

$$P_- =: \exp\left[-\sum_{i\geq 0}\psi_i\psi_i^*\right] : \tag{A1.9}$$

enjoying the following properties

$$P_-\psi_k = \psi_k^* P_- = 0 \ , \ k \geq 0 \tag{A1.10}$$

$$[P_-, \psi_{-k}] = [P_-, \psi_{-k}{}^*] = 0 \ , \ k > 0 \tag{A1.11}$$

$$P_-^2 = P_- \tag{A1.12}$$

Let us calculate the state

$$G_0 P_+ e^{\overline{H}(y)}|n\rangle \tag{A1.13}$$

One can easily check that it vanishes at $n < 0$. Indeed, using (2.8), (2.12) and (A1.6), one can obtain that, at $n < 0$, the state

$$e^{\overline{H}(\overline{t})}|n\rangle = \psi^*_{-n}(\overline{t}) \dots \psi^*_{-1}(\overline{t}) e^{\overline{H}(\overline{t})}|0\rangle$$

contains only negative modes ψ^*_{-m} ($m > 0$). Hence, action of P_+ annihilates this state due to formula (A1.5). At $n \geq 0$, using (2.8), (2.12) and (A1.6), one gets

$$P_+ e^{\overline{H}(\overline{t})}|n\rangle = \psi_{n-1}(\overline{t}) \dots \psi_0(\overline{t}) P_+ e^{\overline{H}(\overline{t})}|0\rangle \tag{A1.14}$$

One should use that

$$P_+ e^{\overline{H}(\overline{t})}|0\rangle = |0\rangle \tag{A1.15}$$

Proof of (A1.15). Denote

$$|\overline{t}\rangle = P_+ e^{\overline{H}(\overline{t})}|0\rangle$$

Then

$$\frac{\partial}{\partial \overline{t}_k}|\overline{t}\rangle = P_+ e^{\overline{H}(\overline{t})} \sum_{i=0}^{k-1} \psi^*_{i-k}\psi_i|0\rangle = 0$$

due to (2.7) and (A1.5). Since $|\overline{t}\rangle \Big|_{\overline{t}_k=0} = |0\rangle$, formula (A1.15) is proved.

Thus, using formulas (A1.8), (A1.9) one obtains

$$\begin{aligned} G_0 P_+ e^{\overline{H}(\overline{t})}|n\rangle &= G\ \psi(\overline{t}) \dots \psi(\overline{t})|0\rangle = \\ = \sum \frac{1}{m!} \int_\gamma \prod_{i=1}^{m} & A(z_i, w_i) dz_i dw_i \psi_+(z_1) \dots \psi_+(z_m) \times \\ \times P_- \psi^*_+(w_m^{-1}) & \dots \psi^*_+(w_1^{-1}) \psi_{n-1}(\overline{t}) \dots \psi_0(\overline{t})|0\rangle \end{aligned} \tag{A1.16}$$

Let us now check that only the $m = n$ term contributes into the infinite sum (A1.16). Indeed, at $m > n$ the state $\psi^*_+(w_m^{-1}) \dots \psi^*_+(w_1^{-1})\psi_{n-1}(\overline{t}) \dots \psi_0(\overline{t})|0\rangle$ vanishes, since, in this case, some of the positive modes in $\psi^*_+(w_i^{-1})$ can be pushed to the vacuum $|0\rangle$ cancelling it. Inversely, at $m < n$ some of the positive modes of $\psi_k(-\overline{y})$ can be pushed to the projector P_- and, due to (A1.10), cancel it. Therefore,

$$\begin{aligned} G_0 P_+ e^{\overline{H}(\overline{t})}|n\rangle = \frac{1}{n!} \int_\gamma \prod_{i=1}^{n} & A(z_i, w_i) dz_i dw_i \psi_+(z_1) \dots \psi_+(z_n) \times \\ \times P_- \psi^*_+(w_n^{-1}) & \dots \psi^*_+(w_1^{-1}) \psi_{n-1}(\overline{t}) \dots \psi_0(\overline{t})|0\rangle \end{aligned} \tag{A1.17}$$

Now we use the following statement:

$$\psi^*_+(w_n^{-1}) \dots \psi^*_+(w_1^{-1})\psi_{n-1}(\overline{t}) \dots \psi_0(\overline{t})|0\rangle = \Delta(w) \exp\left[\sum_{j=1}^{n} \xi(\overline{t}, w_j)\right]|0\rangle \tag{A1.18}$$

Proof of formula (A1.18). Since the number of creation operators (w.r.t. the vacuum $|0\rangle$) $\psi_i(-\overline{y})$ is equal to the number of annihilation operators $\psi^*_+(w_j^{-1})$, after the normal re-ordering

$$\psi^*_+(w_n^{-1}) \dots \psi^*_+(w_1^{-1})\psi_{n-1}(\overline{t}) \dots \psi_0(\overline{t})|0\rangle = const \cdot |0\rangle$$

and

$$const = \langle 0|\psi_+^*(w_n^{-1})\dots\psi_+^*(w_1^{-1})\psi_{n-1}(\bar t)\dots\psi_0(\bar t)|0\rangle =$$
$$= \det\left[\langle 0|\psi_+^*(w_i^{-1})\psi_{j-1}(\bar t)|0\rangle\right]\Big|_{i,j=1,\dots,n}$$

Now, using formulas (2.13), (2.9), one can get

$$\langle 0|\psi_+^*(w_i^{-1})\psi_{j-1}(\bar t)|0\rangle = w_i^{j-1}e^{\xi(\bar t, w_i)}$$

Thus, finally

$$const = \det[w_i^{j-1}e^{\xi(\bar t,w_i)}] = \Delta(w)\exp[\sum_{j=1}^n \xi(\bar t, w_j)]$$

Substituting now (A1.18) into (A1.17) and using the obvious fact that $P_-|0\rangle = |0\rangle$ and $\psi_-(z_i)|0\rangle = 0$, one obtains

$$G_0P_+e^{\overline{H}(\bar t)}|n\rangle = \frac{1}{n!}\int_\gamma \prod_{i=1}^n A(z_i,w_i)e^{\xi(\bar t,w_i)}dz_idw_i\Delta(w)\psi(z_1)\dots\psi(z_n)|0\rangle \tag{A1.19}$$

At last, using one of the main formulas of the paper [43] (which can be easily proved within the bosonization framework)

$$\psi(z_1)\dots\psi(z_n)|0\rangle = \Delta(z)\exp\left[\overline{H}\left(\sum_{i=1}^n \epsilon(z_i)\right)\right]|0\rangle \tag{A1.20}$$

where $\epsilon(z_i)$ is the vector with components $\epsilon_k(z_i) = \dfrac{1}{k}z_i^k$, we obtain the desired result:

$$G_0P_+e^{\overline{H}(\bar t)}|n\rangle = \frac{1}{n!}\int_\gamma \prod_{i=1}^n A(z_i,w_i)e^{\xi(\bar t,w_i)}dz_idw_i\Delta(w)\Delta(z)\exp\left[\overline{H}\left(\sum_{i=1}^n \epsilon(z_i)\right)\right]|0\rangle \tag{A1.21}$$

Thus, we finally get:

$$\tau_n(t,\bar t) = \langle n|e^{H(x)}G_0P_+e^{\overline{H}(\bar t)}|n\rangle = \frac{1}{n!}\int_\gamma \Delta(w)\Delta(z)\prod_{i=1}^n A(z_i,w_i)e^{\xi(t,z_i)+\xi(\bar t,w_i)}dz_idw_i \tag{A1.22}$$

Determinant representation. Using formulas (2.8), (A1.7) and the property $[P_+, g_0] = 0$, one obtains:

$$\tau_n(t,\bar t) = \langle 0|\psi_0^*\dots\psi_{n-1}^*e^{H(t)}G_0P_+e^{\overline{H}(\bar t)}\psi_{n-1}\dots\psi_0|0\rangle =$$
$$= \langle 0|e^{H(t)}\psi_0^*(-t)\dots\psi_{n-1}^*(-t)P_+G_0P_+\psi_{n-1}(\bar t)\dots\psi_0(\bar t)e^{\overline{H}(\bar t)}|0\rangle$$

Since $\psi_i^*(-t)$ and $\psi_i(\bar t)$ contains only positive modes (see (2.11) and (2.12)), one gets from (A1.6) and (A1.15)

$$\begin{gathered}\tau_n(t,\bar t) = \langle 0|\psi_0^*(-t)\dots\psi_{n-1}^*(-t)G_0\psi_{n-1}(\bar t)\dots\psi_0(\bar t)|0\rangle = \\ \det\left[\langle 0|\psi_i^*(-t)G_0\psi_j(\bar t)|0\rangle\right]\big|_{i,j=0,\dots,n-1}\end{gathered} \tag{A1.23}$$

The same arguments we used to derive (A1.17) from (A1.16), when applied to (A1.23), allows one to conclude that only the linear term in $A(z,w)$ contributes. Therefore, using (A1.10), one obtains

$$\begin{gathered}\langle 0|\psi_i^*(-t)G\psi_j(\bar t)|0\rangle = \int_\gamma A(z,w)dzdw\langle 0|\psi_i^*(-t)\psi_+(z)P_-\psi_+^*(w^{-1})\psi_j(\bar t)|0\rangle = \\ = \int_\gamma z^iw^jA(z,w)e^{\xi(t,z)+\xi(\bar t,w)}dzdw = \partial_t^i\partial_{\bar t}^j\int_\gamma A(z,w)e^{\xi(t,z)+\xi(\bar t,w)}dzdw\end{gathered} \tag{A1.24}$$

Thus, the final expression for the τ-function in determinant form is

$$\tau_n(t,\bar t) = \det\left[\partial_{x_1}^i\partial_{\bar t_1}^j\int_\gamma A(z,w)e^{\xi(t,z)+\xi(\bar t,w)}dzdw\right]\Big|_{i,j=0,\dots,n-1} \tag{A1.25}$$

Appendix 2 Fundamental representations of $SL_q(p)$

In this Appendix we generalize the structure of fundamental representations described in s.5.1 to the quantum case, i.e. to the group $SL_q(p)$ with $q \neq 1$ (see [26] and references therein).

The main new notion that we need is q-antisymmetrization that is defined to be the sum over all the permutations

$$([1,\ldots,k]_q) = \sum_P (-q)^{\deg\, P} \left(P(1),\ldots,P(k)\right) \tag{A2.1}$$

where

$$\deg\, P = \text{number of inversions in } P \tag{A2.2}$$

For instance, q-antisymmetrization is used to define the q-determinant:

$$\det_q A \sim A^{[1}_{[1} \ldots A^{p]_q}_{p]_q} = \sum_{P,P'} (-q)^{\deg\, P + \deg\, P'} \prod_a A^{P(a)}_{P'(a)} \tag{A2.3}$$

Note that this quantity does not obligatory coincide with $A^1_{[1} \ldots A^p_{p]_q}$. For instance, for $p = 2$ (A2.3) gives $\frac{1}{[2]}(A^1_1 A^2_2 - qA^1_2 A^2_1 - qA^2_1 A^1_2 + q^2 A^2_2 A^1_1)$, while q-antisymmetrizing over only the lower indices would give merely $A^1_1 A^2_2 - qA^1_2 A^2_1$. Moreover, $A^1_{[1} A^1_{2]} = A^1_1 A^2_2 - qA^1_2 A^1_1$ is not obliged to vanish and even $A^1_{[1} A^1_{1]_q} = (1-q)(A^1_1)^2 \neq 0$.

The "normal" properties of the q-antisymmetrization restore, if one considers as A^i_j elements of the coordinate ring $A(GL(p))$ (algebra of functions on the quantum group $GL_q(p)$). These matrix elements satisfy the following commutation relations (in the particular case of $p = 2$, they reduce to (4.39)) [19]:

$$\begin{gathered}
\forall i, \forall j_1 < j_2 \quad A^i_{[j_1} A^i_{j_2]_q} = 0, \qquad (ab = qba,\ cd = qdc) \\
\forall i_1 < i_2, \forall j \quad A^{[i_1}_j A^{i_2]_q}_j = 0, \qquad (ac = qca,\ bd = qdb) \\
\forall i_1 \neq i_2, j_1 \neq j_2 \quad A^{i_1}_{j_2} A^{i_2}_{j_1} = A^{i_2}_{j_1} A^{i_1}_{j_2}, \qquad (bc = cb) \\
\forall i_1 < i_2,\ j_1 < j_2 \\
A^{i_1}_{j_1} A^{i_2}_{j_2} - A^{i_2}_{j_2} A^{i_1}_{j_1} = (q - q^{-1}) A^{i_1}_{j_2} A^{i_2}_{j_1} \qquad (ad - da = (q - q^{-1})bc)
\end{gathered} \tag{A2.4}$$

For $A \in GL_q(p)$

$$\det_q A = A^1_{[1} \ldots A^p_{p]_q} = A^{[1}_1 \ldots A^{p]_q}_p \tag{A2.5}$$

and $A^{i_1}_{[1} \ldots A^{i_k}_{k]_q} = 0$ if two arbitrary upper indices coincide (but all the lower indices are different: even for $A \in GL_q(p)$ it is still correct that $A^1_{[1} A^1_{1]_q} = (1-q)(A^1_1)^2 \neq 0$).

The notion of q-antisymmetrization is of importance when constructing fundamental representations, since the k-th fundamental representation $F^{(k)}$ of $SL_q(p)$ is the q-skew degree of $F = F^{(1)}$: at $q \neq 1$ we obtain instead of (5.2):

$$F^{(k)} = \left\{ \Psi^{(k)}_{i_1 \ldots i_k} \sim \psi_{[i_1} \ldots \psi_{i_k]_q}, \quad i_1 < \ldots < i_k \right\} \tag{A2.6}$$

Note that now one should *manifestly* require for all i_a to differ.

All the formulas like (5.4) for the intertwining operators remains unchanged provided the antisymmetrization in them is replaced by the q-antisymmetrization (with q-ϵ-symbol defined in the obvious way). Instead of (5.7), one now gets:

$$g^{(k)} \begin{pmatrix} i_1 \ldots i_k \\ j_1 \ldots j_k \end{pmatrix} \sim \det_q g^{i_a}_{j_b}, \qquad i_1 < \ldots i_k, \text{ or } j_1 < \ldots j_k \tag{A2.7}$$

while (5.8) turns into

$$
\begin{aligned}
&g^{(k)}\begin{pmatrix} i_1\dots i_k \\ [j_1\dots j_k \end{pmatrix} g^{(k')}\begin{pmatrix} i'_1\dots i'_{k'} \\ j_{k+1}]_q j'_1\dots j'_{k'-1} \end{pmatrix} = \\
&= g^{(k+1)}\begin{pmatrix} i_1\dots i_k[i'_k \\ j_1\dots j_{k+1} \end{pmatrix} g^{(k'-1)}\begin{pmatrix} i'_1\dots i'_{k'-1}]_q \\ j'_1\dots j'_{k'-1} \end{pmatrix}
\end{aligned}
\tag{A2.8}
$$

$$
\begin{aligned}
&i_1<\dots<i_k, \ \text{ or } \ j_1<\dots j_k, \qquad i'_1<\dots<i'_{k'} \ \text{ or } \ j_{k+1}<j'_1<\dots<j'_{k'-1}, \\
&i_1<\dots<i_k<i'_{k'}, \ \text{ or } j_1<\dots<j_{k+1}, \qquad i'_1<\dots<i'_{k'-1}, \ \text{ or } j'_1<\dots<j'_{k'-1}
\end{aligned}
\tag{A2.9}
$$

Exactly similar to (5.8), these are just an identity for the matrices from $GL_q(p)$.[31] The feature that is essentially new as compared with (5.8) is explicit restricting the indices i, j, i', j', which makes the translation to the language of generating functions more sophisticated.

Let us note that, similar to the classical case, one can construct from the quantum minors (A2.7) local coordinates on the quantum flag space [79]. Similar to the classical case, these coordinates satisfy a set of (quantum) bilinear (Plucker) relations and present some quantum generalization of the Plucker coordinates [80]. Unfortunately, so far no consistent ways to parametrize the Plucker coordinates is known.

Now we would like to discuss formula (A2.6) in detail. We start with the classical case, since the structure of representations remains practically unchanged in the course of quantization.

The Lie algebra $SL(p)$ is generated by the generators $T_{\pm\alpha}$ and Cartan operators H_β so that $[H_\beta, T_{\pm\alpha}] = \pm\frac{1}{2}(\boldsymbol{\alpha\beta})T_{\pm\alpha}$. All the elements of representation are the eigenfunctions of the Cartan generators H_β, $H_\beta|\boldsymbol{\lambda}\rangle = \frac{1}{2}(\boldsymbol{\beta\lambda})|\boldsymbol{\lambda}\rangle$. The highest weight vectors of the representations $F^{(k)}$ – $\boldsymbol{\mu}_k$ – form a system dual to the *simple* roots $\boldsymbol{\alpha}_i$, $i=1,\dots,r$: $(\boldsymbol{\mu}_i\boldsymbol{\alpha}_j) = \delta_{ij}$, and $\boldsymbol{\rho} \equiv \frac{1}{2}\sum_{\alpha>0}\boldsymbol{\alpha} = \sum_i \boldsymbol{\mu}_i$.

The representation $F^{(1)}$ contains the states of the form

$$
\psi_i = T_{-(i-1)}\dots T_{-2}T_{-1}\psi_1, \quad i=1,\dots,p
\tag{A2.11}
$$

Moreover,

$$
T_{-i}\psi_j = \delta_{ij}\psi_{i+1}
\tag{A2.12}
$$

(thus, for $T_- = \sum_{i=1}^r T_{-i}$ $T_-^i\psi_j = \psi_{j+i}$ and one obtains (5.1)), and

$$
\boldsymbol{\lambda}(\psi_i) = \boldsymbol{\mu}_1 - \boldsymbol{\alpha}_1 - \dots - \boldsymbol{\alpha}_{i_1}
\tag{A2.13}
$$

Here $T_{\pm i} \equiv T_{\pm\alpha_i}$ are the generators associated with the simple roots. Denote the corresponding basis in the Cartan algebra $H_i = H_{\alpha_i}$, $H_i|\boldsymbol{\lambda}\rangle = \frac{1}{2}(\boldsymbol{\alpha}_i\boldsymbol{\lambda})|\boldsymbol{\lambda}\rangle = \lambda_i|\boldsymbol{\lambda}\rangle$. Then,

$$
\lambda_i^{(j)} \equiv \lambda_i(\psi_j) = \frac{1}{2}(\delta_{ij} - \delta_{i,j-1})
\tag{A2.14}
$$

This formula, along with (A2.11) and commutation relations of the algebra, gives $||\psi_i||^2 = 1$. Since the co-multiplication in the classical case is just $\Delta(T) = T\otimes I + I\otimes T$, $\psi_{[1}\dots\psi_{k]}$ exhaust the highest weight vectors (i.e. they are cancelled by all $\Delta_k(T_{+i})$ and, thus, by all $\Delta_k(T_{+\alpha})$).

[31] It is important that g belongs to some $GL_q(p)$, i,e, its elements satisfy the proper commutation relations. This is certainly *implied* in derivation of (A2.8), since is assumed that g is the group element. Be it not the case, one would need to understand (A2.7) in the sense of (A2.3) (in particular, one would need to write *and* instead of *or* in (A2.7)), and then one would run in a contradiction with (A2.8). To make it more transparent, let us take $k=k'=1$ so that (A2.8) becomes

$$
g^i_{[j_1}g^{i'}_{j_2]_q} = g^{(2)}\begin{pmatrix} i\ i' \\ j_1 j_2 \end{pmatrix}
\tag{A2.10}
$$

and the l.h.s. of this formula is equal to $g^i_{j_1}g^{i'}_{j_2} - qg^i_{j_2}g^{i'}_{j_1}$, while the r.h.s. would be interpreted as $g^i_{j_1}g^{i'}_{j_2} - qg^i_{j_2}g^{i'}_{j_1} - qg^{i'}_{j_1}g^i_{j_2} + q^2g^{i'}_{j_2}g^i_{j_1}$.

Quantum UEA $U_q(SL(p))$ is given by the Chevalle generators $T_{\pm i}$ corresponding to the simple roots and $q^{\pm H_i}$, with the defining commutation relations (A_{ij} is the Cartan matrix for $SL(p)$) [81, 71]

$$
\begin{gathered}
q^{H_i}T_{\pm j}q^{-H_i} = q^{\pm A_{ij}}T_{\pm j} \\
[T_{+i}, T_{-j}] = \delta_{ij}\frac{q^{2H_i} - q^{-2H_i}}{q - q^{-1}}
\end{gathered}
\tag{A2.15}
$$

and the co-multiplication

$$
\begin{gathered}
\Delta(T_{\pm i}) = q^{H_i} \otimes T_{\pm i} + T_{\pm i} \otimes q^{-H_i} \\
\Delta(q^{\pm H_i}) = q^{\pm H_i} \otimes q^{\pm H_i}
\end{gathered}
\tag{A2.16}
$$

Besides, the Chevalle generators satisfy additional (Serre) relations [81]. One can use, instead of the Chevalle basis, the basis of generators corresponding to all roots. In this case, the Serre identities are no longer needed (they become relations in the algebra). However, in this case the co-multiplication for $T_{\pm\alpha}$, which can be easily read off from (A2.16) turns out to be quite complicated. For instance, one obtains for the generator corresponding to the root $\boldsymbol{\alpha}$ of "weight" 2, i.e. $T_{\pm\alpha} = \pm[T_{\pm\alpha_i}, T_{\pm\alpha_{i+1}}]$,

$$
\begin{gathered}
\Delta(T_{-\alpha}) = -[\Delta(T_{\alpha_i}), \Delta(T_{\alpha_{i+1}})] = q^{H_\alpha} \otimes T_\alpha + T_\alpha \otimes q^{-H_\alpha} + \\
+(q^{1/2} - q^{-1/2})\left[(T_{-\alpha_i} \otimes T_{-\alpha_{i+1}})(q^{H_{i+1}} \otimes q^{-H_i}) - (T_{-\alpha_{i+1}} \otimes T_{-\alpha_i})(q^{H_i} \otimes q^{-H_{i+1}})\right]
\end{gathered}
\tag{A2.17}
$$

With the co-multiplication given, one can easily check formula (A2.6). For instance, for $F^{(2)}$:

$$
\Delta(T_{+i})(\psi_1\psi_2 - q\psi_2\psi_1) = \delta_{i,1}(q^{\lambda_1^{(1)}}\psi_1\psi_1 - q^{1-\lambda_1^{(1)}}\psi_1\psi_1) = 0 \tag{A2.18}
$$

since $\lambda_1^{(1)} = \frac{1}{2}$. Thus, $\Psi_{12}^{(2)} \equiv \psi_{[1}\psi_{2]_q}$ is actually the highest weight vector. Similarly, ($i < j$):

$$
\begin{gathered}
\Delta(T_{-l})\Psi_{ij}^{(2)} = \Delta(T_{-l})(\psi_i\psi_j - q\psi_j\psi_i) = \\
= \delta_{li}q^{-\lambda_i^{(j)}}(\psi_{i+1}\psi_j - q^{1+2\lambda_i^{(j)}}\psi_j\psi_{i+1}) + \delta_{lj}q^{\lambda_j^{(i)}}(\psi_i\psi_{j+1} - q^{1-2\lambda_j^{(i)}}\psi_{j+1}\psi_i)
\end{gathered}
\tag{A2.19}
$$

In accordance with (A2.14), for $i < j$ always $\lambda_j^{(i)} = 0$, while $\lambda_i^{(j)} = 0$ unless $j = i+1$, and $\lambda_i^{(i+1)} = -\frac{1}{2}$. Thus,

$$
\Delta(T_{-l})\Psi_{ij}^{(2)} = \delta_{lj}\Psi_{i,j+1}^{(2)} + \delta_{li}\Psi_{i+1,j}^{(2)}(1 - \delta_{i+1,j}) \tag{A2.20}
$$

This allows one to determine the action of any $\Delta(T_{-\alpha})$. One can easily describe the action of all $\Delta(T_{+i})$ and repeat the procedure for other representations $F^{(k)}$.

Appendix 3 Manifest example: τ_2 for $SL_q(p)$

In this Appendix, we consider the simplest τ-function for the group $SL_q(p)$ in the simplest non-trivial (second fundamental) representation [29]. Denote through $\{u\}$ and $\{v\}$ the subsets $\{s\}$ such that $i(s) = 1$ and $i(s) = 2$ respectively. Then,

$$
\begin{gathered}
\tau_2 = \tau_1^{00}(\{q\xi_u\}, \{q^{-1}\xi_v\}, \xi_s;\ \{\bar\xi_u\}, \{\bar\xi_v\}, \bar\xi_s)\tau_1^{11}(\{\xi_u\}, \{\xi_v\}, \xi_s;\ \{q\bar\xi_u\}, \{\bar\xi_v\}, \bar\xi_s) - \\
-q\tau_1^{01}(\{q\xi_u\}, \{q^{-1}\xi_v\}, \xi_s;\ \{\bar\xi_u\}, \{\bar\xi_v\}, \bar\xi_s)\tau_1^{10}(\{\xi_u\}, \{\xi_v\}, \xi_s;\ \{q^{-1}\bar\xi_u\}, \{q\bar\xi_v\}, \bar\xi_s) - \\
-q\tau_1^{10}(\{q^{-1}\xi_u\}, \{\xi_v\}, \xi_s;\ \{\bar\xi_u\}, \{\bar\xi_v\}, \bar\xi_s)\tau_1^{01}(\{\xi_u\}, \{\xi_v\}, \xi_s;\ \{q\bar\xi_u\}, \{\bar\xi_v\}, \bar\xi_s) + \\
+q^2\tau_1^{11}(\{q^{-1}\xi_u\}, \{\xi_v\}, \xi_s;\ \{\bar\xi_u\}, \{\bar\xi_v\}, \bar\xi_s)\tau_1^{00}(\{\xi_u\}, \{\xi_v\}, \xi_s;\ \{q^{-1}\bar\xi_u\}, \{q\bar\xi_v\}, \bar\xi_s)
\end{gathered}
\tag{A3.1}
$$

Here ξ_s denotes all times with $i(s) > 2$. Consider now parametrization A. In formula (A3.1), each set of $\{u\}$ and $\{v\}$ consists of a single element: $\{u\} = \{s = 1\}$, $\{v\} = \{s = 2\}$. Then,

$$\begin{aligned}\tau_2 = \tau_1^{00}(q\xi_1, q^{-1}\xi_2, \xi_i;\ \overline{\xi}_1, \overline{\xi}_2, \overline{\xi}_i)\tau_1^{11}(\xi_1, \xi_2, \xi_i;\ q\overline{\xi}_1, \overline{\xi}_2, \overline{\xi}_i)-\\ -q\tau_1^{01}(q\xi_1, q^{-1}\xi_2, \xi_i;\ \overline{\xi}_1, \overline{\xi}_2, \overline{\xi}_i)\tau_1^{10}(\xi_1, \xi_2, \xi_i;\ q^{-1}\overline{\xi}_1, q\overline{\xi}_2, \overline{\xi}_i)-\\ -q\tau_1^{10}(q^{-1}\xi_1, \xi_2, \xi_i;\ \overline{\xi}_1, \overline{\xi}_2, \overline{\xi}_i)\tau_1^{01}(\xi_1, \xi_2, \xi_i;\ q\overline{\xi}_1, \overline{\xi}_2, \overline{\xi}_i)+\\ +q^2\tau_1^{11}(q^{-1}\xi_1, \xi_2, \xi_i;\ \overline{\xi}_1, \overline{\xi}_2, \overline{\xi}_i)\tau_1^{00}(\xi_1, \xi_2, \xi_i;\ q^{-1}\overline{\xi}_1, q\overline{\xi}_2, \overline{\xi}_i)\end{aligned} \tag{A3.2}$$

This expression can be rewritten in a more compact form using the operators

$$\mathcal{D}_1^L \equiv M_1^- D_1 \otimes I, \quad \mathcal{D}_1^R \equiv M_1^+ M_2^- \otimes D_1, \quad \overline{\mathcal{D}}_1^L \equiv \overline{D}_1 \otimes \overline{M}_1^- \overline{M}_2^+, \quad \overline{\mathcal{D}}_1^R \equiv I \otimes \overline{M}_1^+ \overline{D}_1$$

$$\tau_2 = \left(\mathcal{D}_1^R\overline{\mathcal{D}}_1^R - q\mathcal{D}_1^L\overline{\mathcal{D}}_1^R - q\mathcal{D}_1^R\overline{\mathcal{D}}_1^L + q^2\mathcal{D}_1^L\overline{\mathcal{D}}_1^L\right)\tau_1 \otimes \tau_1 = \left(\mathcal{D}_1^R - q\mathcal{D}_1^L\right) \cdot \left(\overline{\mathcal{D}}_1^R - q\overline{\mathcal{D}}_1^L\right)\tau_1 \otimes \tau_1 \tag{A3.3}$$

These operators satisfy the commutation relations that form the algebra similar to that generated by θ and χ in (4.56): $\mathcal{D}_1^L\mathcal{D}_1^R = q\mathcal{D}_1^R\mathcal{D}_1^L$, $\overline{\mathcal{D}}_1^L\overline{\mathcal{D}}_1^R = q\overline{\mathcal{D}}_1^R\overline{\mathcal{D}}_1^L$. Formula for τ_2 can be rewritten in a more "invariant" form in terms of the operators

$$\mathbb{D}_i^L \equiv D_i \otimes I, \quad \mathbb{D}_i^R \equiv \prod_j M_j^{-\vec{\alpha}_i\vec{\alpha}_j} \otimes D_i, \quad \overline{\mathbb{D}}_i^L \equiv \overline{D}_i \otimes \prod_j \overline{M}_j^{-\vec{\alpha}_i\vec{\alpha}_j}, \quad \overline{\mathbb{D}}_i^R \equiv I \otimes \overline{D}_i \tag{A3.4}$$

that commute as

$$\mathbb{D}_i^L\mathbb{D}_j^R = q^{\vec{\alpha}_i\vec{\alpha}_j}\mathbb{D}_j^R\mathbb{D}_i^L, \qquad \overline{\mathbb{D}}_i^L\overline{\mathbb{D}}_j^R = q^{\vec{\alpha}_i\vec{\alpha}_j}\overline{\mathbb{D}}_j^R\overline{\mathbb{D}}_i^L \tag{A3.5}$$

Indeed, one can write

$$\tau_2 = M_1^- \otimes \overline{M}_1^+ \left(\mathbb{D}_1^R - q\mathbb{D}_1^L\right) \cdot \left(\overline{\mathbb{D}}_1^R - q\overline{\mathbb{D}}_1^L\right)\tau_1 \otimes \tau_1 \tag{A3.6}$$

Appendix 4 Construction of the Whittaker function for $SL(p)$

In this Appendix, we construct the Toda WF corresponding to the group $SL(p)$ [30]. We start with notations and definitions. The standard textbooks for us are [82].

A4.1 Notations

Lie algebra of the Lie group $SL(p)$ is completely given by the following commutation relations of the generators corresponding to the simple roots (Chevalle generators):

$$[T_{\pm i}, T_{0,j}] = \mp A_{ij}T_{\pm,i}, \quad [T_{+i}, T_{-j}] = \delta_{ij}T_{0,j}, \quad i,j = 1,\ldots,p-1 \tag{A4.1}$$

and additionally the Serre relations

$$\mathrm{ad}_{T_{\pm i}}^{1-A_{ij}}(T_{\pm j}) = 0 \tag{A4.2}$$

where $\mathrm{ad}_x^k(y) \equiv \underbrace{[x,[x,...,[x,y]..]]}_{k\ \text{times}}$. All other commutation relations can be obtained from (A4.1)-(A4.2), the generators which correspond to positive (negative) nonsimple roots being constructed from the positive (negative) simple root (Chevalle) generators by the manifest formula $[T_{\boldsymbol{\alpha}}, T_{\boldsymbol{\beta}}] = N_{\boldsymbol{\alpha},\boldsymbol{\beta}}T_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$. Here the generator $T_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ corresponds to the non-simple root $\boldsymbol{\alpha}+\boldsymbol{\beta}$ and $N_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ are some

non-zero structure constants. Making use of nonsimple root generators, the Serre identities are replaced by appropriate Lie algebra relations.

We also use the following notations: $\boldsymbol{\alpha}_i$ are the simple roots in the corresponding Lie algebra, $A_{ij} \equiv \boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j$ is the Cartan matrix, $\boldsymbol{\mu}_i$ are the fundamental weights that, by definition, lie in the dual lattice $\boldsymbol{\mu}_i \cdot \alpha_j = \delta_{ij}$ (i.e. $\boldsymbol{\mu}_i = A_{ij}^{-1}\boldsymbol{\alpha}_j$) and $\boldsymbol{\rho} \equiv \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\Delta^+}\boldsymbol{\alpha} = \sum_i \boldsymbol{\mu}_i$, where Δ^+ is the set of all the positive roots. In these notations, $\phi_i = -\boldsymbol{\mu}_i \cdot \boldsymbol{\phi}$. Roots $\boldsymbol{\alpha}_i$ can be also considered as vectors in the p-dimensional Cartan plane of the group $GL(p)$: $\boldsymbol{\alpha}_i = \boldsymbol{e}_{i+1} - \boldsymbol{e}_i$, $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$.

In order to define the Casimir operators, let us fix a representation ρ of UEA $U(SL(p))$. Define the L-operator by the formula [19]

$$L \equiv \sum_{\boldsymbol{\alpha}\in\Delta} \rho(T_{\boldsymbol{\alpha}}) \otimes T_{-\boldsymbol{\alpha}} + \sum_i A_{ij}^{-1}\rho(T_i) \otimes T_j \tag{A4.3}$$

Then the k-th Casimir operator can be defined via the formula

$$C_k \equiv \mathrm{tr}_\rho L^k \tag{A4.4}$$

where the trace is taken over the representation ρ. Indeed, since the result does not depend on the representation ρ, one can choose it to be the simplest one – the first fundamental representation. Then, the Casimir operator can be easily calculated. In particular, the quadratic Casimir acquires the form

$$C_2 = \sum_{\boldsymbol{\alpha}\in\Delta} T_{\boldsymbol{\alpha}} T_{-\boldsymbol{\alpha}} + \sum_{ij}^{p-1} A_{ij}^{-1} T_{0,i} T_{0,j} \tag{A4.5}$$

where the first sum goes over all (positive and negative) roots.

A4.2 Representations

For arbitrary group $SL(p)$, one can define the (right) regular representation only in general terms of the group acting on the algebra of functions:

$$\pi_{reg}(h) f(g) = f(gh) \tag{A4.6}$$

Hence, we mainly use the group (not algebraic) terms (see, however, [30]). It turns out that, for the generic group $SL(p)$, one can describe restricting the space of functions onto irreducible representations. To this end, one needs to consider the representations induced by one-dimensional representations of the Borel subgroup. This restricts the space of all functions onto the functions satisfying the covariantness condition

$$f_\lambda(bg) = \chi_\lambda(b) f_\lambda(g) \tag{A4.7}$$

where b is an element of the Borel subgroup of the lower-triangle matrices and χ_λ is the character of the Borel subgroup, which is equal to

$$\chi_\lambda(b) = \prod_{i=1}^{p-1} \mid b_{ii} \mid^{(\boldsymbol{\lambda}-\boldsymbol{\rho})\boldsymbol{e}_i} (\mathrm{sign} b_{ii})^{\epsilon_i} \tag{A4.8}$$

where ϵ_i are equal either to 0 or to 1. For the sake of simplicity, we consider only the representations without sign factors, although the generic case can be also easily treated. The described representations belong to the principal sphere series.

Thus, the representations are given by restricting the space of functions onto the functions defined on the quotient group $B\backslash G$ that can be identified with the (nilpotent) subgroup of the

strictly upper-triangle matrices N_+. Given λ, there is the natural Hermitian bilinear form on the space of matrix elements X, which is just given by the flat measure

$$< f_L|f_R >_\lambda = \int_{X=B\backslash G} \overline{f_{L,\lambda}(x)} f_{R,\lambda}(x) \prod_{ij} dx_{ij} \tag{A4.9}$$

This form becomes the scalar product when λ is pure imaginary, which leads to unitary irreducible representations of the principal series.

A4.3 Hamiltonian and Schroedinger equation

We consider the following reduction conditions analogous to (6.30):

$$T_{+,i}|\psi_R >= i\mu_i^R|\psi_R > \tag{A4.10}$$

and

$$< \psi_L|T_{-,i} = i\mu_i^L < \psi_L| \tag{A4.11}$$

where $\mu_i^{R,L}$ are cosmological constants. These conditions are completely given by the Chevalle generators, since all others are easily obtained from the commutation relations. More precisely, the action of nonsimple root generators just vanishes. Now one can easily get

$$\begin{gathered}(\boldsymbol{\lambda}^2-\boldsymbol{\rho}^2)F^{(\boldsymbol{\lambda})}(\boldsymbol{\phi}) \equiv (\boldsymbol{\lambda}^2-\boldsymbol{\rho}^2) < \psi_L|e^{-\boldsymbol{\mu}_i\boldsymbol{\phi}T_{0,i}}|\psi_R >=< \psi_L|e^{-\boldsymbol{\mu}_i\boldsymbol{\phi}T_{0,i}}C_2|\psi_R >= \\ =< \psi_L|e^{-\boldsymbol{\mu}_i\boldsymbol{\phi}T_{0,i}}\left(2\sum_{\boldsymbol{\alpha}\in\Delta^+}T_{\boldsymbol{\alpha}}T_{-\boldsymbol{\alpha}}+2\sum_{ij}A_{ij}^{-1}T_{0,j}+\sum_{ij}A_{ij}^{-1}T_{0,i}T_{0,j}\right)|\psi_R >= \\ =\left(\frac{\partial^2}{\partial\boldsymbol{\phi}^2}+2\sum_i\partial(\boldsymbol{\alpha}_i\boldsymbol{\phi})-2\sum_i\mu_i^L\mu_i^R e^{\boldsymbol{\alpha}_i\boldsymbol{\phi}}\right)< \psi_L|e^{-\boldsymbol{\mu}_i\boldsymbol{\phi}T_{0,i}}|\psi_R >\end{gathered} \tag{A4.12}$$

and $\Psi^{(\boldsymbol{\lambda})}(\boldsymbol{\phi}) = e^{-\boldsymbol{\rho\phi}}F^{(\boldsymbol{\lambda})}(\boldsymbol{\phi})$ satisfies the Toda chain Schroedinger equation

$$\left(\frac{\partial^2}{\partial\boldsymbol{\phi}^2}-2\sum_i\mu_i^L\mu_i^R e^{\boldsymbol{\alpha}_i\boldsymbol{\phi}}\right)\Psi^{(\boldsymbol{\lambda})}(\boldsymbol{\phi}) = \boldsymbol{\lambda}^2\Psi^{(\boldsymbol{\lambda})}(\boldsymbol{\phi}) \tag{A4.13}$$

A4.4 Solution to the Schroedinger equation

In order to calculate the Whittaker function explicitly, one should manifestly construct solutions to the conditions (A4.10) and (A4.11). However, we do not use here the algebraic approach, i.e. do not solve the corresponding differential equation[32], but, instead, use the group theory approach. That is, we find the functions $f_{L,\lambda}(g)$ and $f_{R,\lambda}(g)$ in the representation space, which satisfy the conditions

$$\begin{gathered}\pi_\lambda(z)f_{R,\lambda}(g) = f_{R,\lambda}(gz) = e^{\sum i\mu_i z_{i,i+1}}f_{R,\lambda}(g) = e^{iTr(\mu z)}f_{R,\lambda}(g), \\ (\mu)_{ij} = \delta_{i-1,j}\mu_i, \quad g,z\in N_+\end{gathered} \tag{A4.14}$$

and

$$\pi_\lambda(z^t)f_{L,\lambda}(g) = e^{iTr(\mu_L z)}f_{L,\lambda}(g) \tag{A4.15}$$

which are the group counterparts of the conditions (A4.10) and (A4.11).

Solving the first condition is equivalent to constructing a one-dimensional representation of the group of the upper-triangle matrices. At first, we construct an additive character of the group using the fact that, in the product of two upper-triangle matrices, the elements lying on the

[32]This is done in [30].

next-to-main diagonal are summed. Then, one can exponentiate this character and obtain the one-dimensional representation we are looking for:

$$f^{\mu}_{R,\lambda}(x) = e^{i\mathrm{tr}\mu x} \tag{A4.16}$$

In order to find the function $f_{L,\lambda}(x)$ that satisfies A4.15), we use the inner automorphism of the group $SL(p)$, which maps strictly upper-triangle matrices into strictly lower-triangle ones. This automorphism is explicitly described by the matrix

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \ldots & 0 & 0 & 0 & 1 & 0 \\ 0 & \ldots & 0 & 0 & 1 & 0 & 0 \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ & \ldots & 1 & 0 & 0 & \ldots & \\ 0 & 1 & 0 & \ldots & 0 & \ldots & 0 \\ 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{A4.17}$$

i.e.

$$S_{ij} = \delta_{i+j,p+1} \tag{A4.18}$$

and

$$(S^{-1}zS)_{ij} = z_{p+1-i,p+1-j} \tag{A4.19}$$

In fact, it is necessary for the matrix S to be an element of the group $SO(p)$ (generally it is constructed as an element of the Weyl group [30]). Therefore, it should be normalized so that it has the unit determinant. One can do this multiplying the matrix by -1, which does not effect formula (A4.19). Note that the automorphism S maps the element $z_{i,i-1}$ into the element $z_{p-i,p+1-i}$ instead of $z_{i-1,i}$ (because of the condition (A4.11), we are only interested in the elements z lying on the next-to-main diagonal). The situation can be corrected by the proper reflection of the matrix μ. Namely, one should introduce the new matrix $\widetilde{\mu}_L \equiv \mu^L_{p-i}\delta_{i-1,j}$ so that the condition (A4.15) can be rewritten as

$$\pi_\lambda(SzS^{-1})f_L(g) = f_L(gSzS^{-1})e^{iTr(\widetilde{\mu}_L z)}f_{L,\lambda}(gS^{-1}S) \tag{A4.20}$$

i.e.

$$f_L(gS^{-1}z) = e^{iTr(\widetilde{\mu}_L z)}f_{L,\lambda}(gS^{-1}) \tag{A4.21}$$

We already know that the solution to this equation in upper-triangle matrices is

$$f_L(gS^{-1})\Big|_{B_-=0} = f_L(n_+) = e^{iTr(\widetilde{\mu}_L n_+)} \tag{A4.22}$$

where n_+ the upper-triangle part of gS^{-1}. We need, however, the solution of (A4.20) for $g = x \in N_+$, i.e. some re-calculation is necessary:

$$f_L(xS^{-1}) = \chi_\lambda(xS^{-1})f_L(n_+) = \chi_\lambda(xS^{-1})e^{iTr(\widetilde{\mu}_L n_+)} \tag{A4.23}$$

Let us calculate this function more explicitly. First of all, elements of the diagonal matrix h in the Gauss decomposition of any matrix g are given by the formula[33]

$$h_i = \frac{\Delta_i(g)}{\Delta_{i-1}(g)}, \quad h_1 = \Delta_1(g), \quad \Delta_p(g) = \frac{\Delta_1(g)}{\Delta_{p-1}(g)} \tag{A4.24}$$

[33]The simplest way to derive this formula is to note that the diagonal element g_{ii} of the matrix g depends only on matrix elements lying in the left upper corner of the $i \times i$ submatrix. Then, one may use the induction over the rank of matrix to obtain formula (A4.24).

where $\Delta_i(g)$ denotes the upper main $i \times i$ minor of the matrix g. Thus, the diagonal part of xS^{-1} is given by the ratio $h_i = \frac{\Delta_i(xS^{-1})}{\Delta_{i-1}(xS^{-1})}$ and

$$\chi_\lambda(xS^{-1}) = \prod_i \Delta_i^{(\boldsymbol{\lambda}\boldsymbol{\alpha}_i - 1)}(xS^{-1}) \tag{A4.25}$$

In order to get the formula for the elements $(n_+)_{k-1,k}$, note that they depend only on the $k \times k$ submatrix placed in the upper left corner. Consider the element $(k-1,k)$ of the matrix $b^{-1} \equiv n_+ g^{-1}$, $g = xS^{-1}$. It is equal to zero by definition of b and, therefore, one obtains

$$(n_+)_{k-1,k}(g^{-1})_{k,k} + (n_+)_{k-1,k-1}(g^{-1})_{k-1,k} = 0 \tag{A4.26}$$

Using the manifest representation for elements of the inverse matrix, one gets the result:

$$(n_+)_{k-1,k} = \frac{\Delta_{k-1,k}(xS^{-1})}{\Delta_{k-1}(xS^{-1})} \tag{A4.27}$$

where $\Delta_{k-1,k}(xS^{-1})$ is defined to be the determinant of the $(k-1) \times (k-1)$ submatrix of the matrix xS^{-1} with the k-th and $k-1$-th columns exchanged.

Thus, we have the manifest expressions for the functions f_L and f_R. In order to calculate the Whittaker function, one only needs now to determine the action of the Cartan part of the group element g on $f_R(x)$. Note that, although the element xh with $x \in X$ and $h \in H$ does not belong to X, the element $h^{-1}xh$ does. Hence, using (A4.6), one can obtain

$$\pi_\lambda(h) f_{R,\lambda}(x) = f_{R,\lambda}(hh^{-1}xh) = h^{\boldsymbol{\lambda}-\boldsymbol{\rho}} f_{R,\lambda}(h^{-1}xh) \tag{A4.28}$$

Then, using this formula and the manifest expression (A4.16), we get

$$\begin{aligned}\pi_\lambda\left(e^{-\boldsymbol{\mu}_i\boldsymbol{\phi}T_{0,i}}\right) f_{R,\lambda}(x) &= e^{\{-\boldsymbol{\mu}_i\cdot\boldsymbol{\phi}\}\{(\boldsymbol{\lambda}-\boldsymbol{\rho})\cdot(\boldsymbol{e}_{i+1}-\boldsymbol{e}_i)\}} f_{R,\lambda}(e^{\boldsymbol{\mu}_i\boldsymbol{\phi}T_{0,i}} x e^{-\boldsymbol{\mu}_i\boldsymbol{\phi}T_{0,i}}) = \\ &= e^{(\boldsymbol{\rho}-\boldsymbol{\lambda})\boldsymbol{\phi}} f_{R,\lambda}(e^{\boldsymbol{\mu}_i\boldsymbol{\phi}T_{0,i}} x e^{-\boldsymbol{\mu}_i\boldsymbol{\phi}T_{0,i}}) = e^{(\boldsymbol{\rho}-\boldsymbol{\lambda})\boldsymbol{\phi}} e^{i\mathrm{tr} x\mu e^{\boldsymbol{\alpha}\boldsymbol{\phi}}}\end{aligned} \tag{A4.29}$$

where the combination $\mu e^{\boldsymbol{\alpha}\boldsymbol{\phi}}$ denotes the matrix with the matrix elements $\delta_{i-1,j}\mu_i e^{\boldsymbol{\alpha}_i\boldsymbol{\phi}}$. Collecting all this together, we finally obtain for the Whittaker function (6.42):

$$\Psi(\phi) = e^{-\boldsymbol{\lambda}\boldsymbol{\phi}} \int_{X=B\backslash G} \prod_{i<j} dx_{ij} \prod_{i=1}^{p-1} \Delta_i^{-(\boldsymbol{\lambda}\boldsymbol{\alpha}_i+1)}(xS^{-1}) \times e^{i\mu_i^R x_{i,i+1} e^{\boldsymbol{\alpha}_i\boldsymbol{\phi}} - i\mu_{p-i}^L \frac{\Delta_{i,i+1}(xS^{-1})}{\Delta_i(xS^{-1})}} \tag{A4.30}$$

References

[1] A.Marshakov, "Exact non-perturbative solutions of quantum theory and integrable systems", thesis for doctor of sciences, Lebedev Physical Institute, Moscow, 1997

[2] A.Mironov, "τ-functions and matrix models", thesis for doctor of sciences, Lebedev Physical Institute, Moscow, 1997

[3] A.B.Zamolodchikov, *Higher integrals of motion in two-dimensional field theory models with broken conformal symmetry*, Pis'ma v ZhETF, **46** (1987) 129-132
B.A.Kupershmidt, P.Mathieu, *Quantum KdV like equations and perturbed conformal field theories*, Phys.Lett., **B227** (1989) 245
T.Eguchi, S.K.Yang, *Deformation of conformal field theories and soliton equations*, Phys.Lett., **B244** (1989) 373-378

[4] R.Sasaki, I.Yamanaka, *Virasoro algebra, vertex operators, quantum sine-Gordon and solvable quantum field theories*, Adv.Stud. in Pure Math., **271-296** (1988) 271-296
A.B.Zamolodchikov, *Integrable field theory from conformal field theory*, Adv.Stud. in Pure Math., **19** (1989) 641-674

[5] A.Morozov, *String theory – what it is*, Sov.Phys.Uspekhi, **162** #8 (1992) 84
V.Knizhnik, *Multiloop amplitudes in theory of quantum strings and complex geometry*, Sov.Phys.Uspekhi, **159** #3 (1989) 401-453

[6] S.Saito, *String theories and Hirota's bilinear difference equation*, Phys.Rev.Lett., **59** (1987) 1798
S.Saito, *String amplitudes as solutions to soliton equations*, Phys.Rev., **D36** (1987) 1819

[7] A.Morozov, *String theory and the structure of universal module space*, Phys.Lett., **B196** (1987) 325

[8] E.Barouch, B.M.McCoy, T.T.Wu, Phys.Rev.Lett., **31** (1973) 1409
C.A.Tracy, B.M.McCoy, Phys.Rev.Lett., **31** (1973) 1500
T.T.Wu, B.M.McCoy, C.A.Tracy, E.Barouch, *Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region*, Phys.Rev., **B13** (1976) 316-374

[9] A.R.Its, A.G.Izergin, V.E.Korepin, N.A.Slavnov, *Differential equations for quantum correlation functions*, Int.J.Mod.Phys., **B4** (1990) 1003-1037

[10] M.Jimbo, T.Miwa, M.Sato, *Holonomic quantum fields*, I-V (Publ.RIMS, Kyoto Univ.), **14** (1978) 223; **15** (1979) 201, 577, 871, 1531

[11] D.Bernard, A.LeClair, *Differential Equations for Sine-Gordon Correlation Functions at the Free Fermion Point,* Preprint CLNS 94/1276, SPhT-94-021; hepth/9402144

[12] A.G.Izergin, D.A.Coker, V.E.Korepin, *Determinant formula for the six-vertex model*, J.Phys.: Math.Gen., **A25** (1992) 4315-4334

[13] A.Morozov, *Integrability and matrix models*, Sov.Phys.Uspekhi, **164** #1 (1994) 3-62

[14] A.Marshakov, *Integrable structures in matrix models and physics of 2d gravity*, Int.J.Mod.Phys., **A8** #22 (1993) 3831-3882
A.Mironov, 2*d gravity and matrix models. I.* 2*d gravity*, Int.J.Mod.Phys., **A9** (1994) 4355-4405

[15] N.Seiberg, E.Witten, *Electric-magnetic duality, monopole condensation and confinement in N=2 supersymmetric Yang-Mills theory*, Nucl.Phys., **B426** (1994) 19-52
N.Seiberg, E.Witten, *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, Nucl.Phys., **B431** (1994) 484-550

[16] A.Gorsky, I.Krichever, A.Marshakov, A.Mironov, A.Morozov, *Integrability and exact Seiberg-Witten solution*, Phys.Lett., **B355** (1995) 466-474

[17] A.Mironov, "WDVV equations in Seiberg-Witten theory and associative algebras", hep-th/9704205

[18] A.Gorsky, S.Gukov, A.Mironov "Multiscale $N = 2$ SUSY field theories, integrable systems and their stringy/brane origin – I, II", hepth/9707120; hepth/9710239

[19] N.Yu.Reshetikhin, L.A.Takhtadjan, L.D.Faddeev, *Quantization of Lie groups and Lie algebras*, Algebra and Analysis, **1** (1989) 178-206

[20] L.A.Takhtadjan, L.D.Faddeev, *Quantum inverse scattering method and XYZ Heisenberg model*, Sov.Math.Uspekhi, **34** (1979) 13-63
V.E.Korepin, N.M.Bogoliubov, A.G.Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, 1993
R.J.Baxter, *Exactly solved models in statistical mechanics*, Academic Press, London-New York, 1989
L.D.Faddeev, *Integrable models in 1+1-dimensional quantum field theory*, in: Les Houches lectures 1982, Elsevier, Amsterdam 1984, pp.563-608
E.Sklyanin, *Quantum inverse scattering method. Selected topics*, in: "Quantum group and quantum integrable systems" (Nankai lectures in mathematical physics), ed. Mo-Lin Ge, World Scientific, Singapore 1992, pp.63-97

[21] F.Smirnov, *What are we quantizing in integrable field theory*, hepth/9307097
O.Babelon, D.Bernard, F.Smirnov, *Form factors, KdV and deformed hyperelliptic curves*, Nucl.Phys.Proc.Suppl., **58** (1997) 21-33
R.Kashaev, N.Reshetikhin, *Affine Toda field theory as a 3-dimensional integrable system*, Comm.Math.Phys., **188** (1977) 251-266

[22] S.Kharchev, A.Marshakov, A.Mironov, A.Morozov, *Generalized Kazakov-Migdal-Kontsevich Model: group theory aspects,* Int.J.Mod.Phys., **A10** (1995) 2015-2052

[23] K.Wilson, *Confinement of quarks*, Phys.Rev., **D10** (1974) 2445

[24] G.Weight, *On Operator Constructions in Bosonic Liouville Quantum Gravity,* in: Proceedings of the 28th International Symposium Ahrenshoop, Wendisch-Rietz, Germany, September 1993 and references therein

[25] A.Mironov, A.Morozov, L.Vinet, *On a c-number quantum τ-function*, Theor.Math.Phys., **100** (1994) 119-132

[26] A.Gerasimov, S.Khoroshkin, D.Lebedev, A.Mironov, A.Morozov, *Generalized Hirota equations and representation theory. I. The case of $SL(2)$ and $SL_q(2)$*, Int.J.Mod.Phys., **A10** (1995) 2589-2614

[27] A.Mironov, *Quantum deformation of τ-functions: Bilinear identities and representation theory*, in: CRM Proceedings & Lecture Notes, **9** (1996) 219-237

[28] S.Kharchev, A.Mironov, A.Morozov, 1994 *τ-Functions: Evolution and Determinant Representations*, in: Proceedings of the XXVIII International Symposium Arenshoop, 1995, p.80-89

[29] S.Kharchev, A.Mironov, A.Morozov, 1995 *Non-Standard KP Evolution and Quantum τ-function*, Theor.Math.Phys., **104** (1995) 129-143

[30] A.Gerasimov, S.Kharchev, A.Marshakov, A.Mironov, A.Morozov, M.Olshanetsky, *Liouville type models in group theory framework. I. Finite-dimensional algebras*, preprint FIAN/TD-18/95, ITEP M4/TH-7/95; hep-th/9601161

[31] A.Mironov, *Group theory structures underlying integrable systems*, preprint FIAN/TD-12/96, ITEP/TH-24/96; hep-th/9607123; to appear in Proc. 2nd International A.D.Sakharov Conference

[32] M.Olshanetsky, A.Perelomov, *Explicit solutions of the classical generalized Toda model*, Invent. Math., **54** (1979) 261-269
B.Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. Math., **34** (1979) 195-338

M.Olshanetsky, A.Perelomov, *Toda chain as a reduced system*, Theor.Math.Phys., **45** (1980) 3-18

[33] N.Hurt, *Geometric quantization in action*, D.Riedel Publishing Company, 1983

[34] H.P.B.Jacquet, *Fonctions de Whittaker associees aux groupes de Chevalley*, Bull.Soc.Math. France, **95** (1967) 243-309
G.Schiffmann, *Integrales d'entrelacement et fonctions de Whittaker*, Bull.Soc.Math.France, **99** (1971) 3-72
B.Kostant, *Whittaker vectors and representation theory*, Invent. Math., **48** (1978) 101-184
M.Hashizume, *Whittaker functions on semisimple Lie groups*, Hiroshima Math. Journ., **12** (1982) 259-293

[35] S.G.Gindikin, F.I.Karpelevich, *Plancherel measure of Riemann symmetric spaces of non-positive curvature*, Dokl.Acad.Nauk, **145** (1962) 252-255
S.G.Gindikin, F.I.Karpelevich, *About an integral related to Riemann symmetric spaces of nonpositive curvature*, Izv.Akad.Nauk SSSR, **30** (1966) 1147-1156

[36] M.Semenov-Tian-Shansky, *Quantization of the open Toda chains*, in "Sovremenie problemi matematiki" VINITI, **16** (1987) 194-226

[37] A.Gerasimov, A.Marshakov, A.Morozov, M.Olshanetsky, S.Shatashvili, *Wess-Zumino-Witten model as a theory of free fields,* Int.J.Mod.Phys., **A5** (1990) 2495-2589

[38] A.Zamolodchikov, Al.Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, hep-th/9506136

[39] H.Dorn, H.-J.Otto, *On correlation functions for non-critical strings with $c \leq 1$ $d \geq 1$*, Phys.Lett., **B291** (1992) 39-43
H.Dorn, H.-J.Otto, *Two and three point functions in Liouville theory*, Nucl.Phys., **B429** (1994) 375-388

[40] M.Kashiwara, T.Miwa, *Transformation group for soliton equations. I. The τ-function of the Kadomtsev-Petviashvili equation*, Proc.Japan Acad., **A57** (1981) 342-347
E.Date, M.Kashiwara, T.Miwa, *Transformation group for soliton equations. II. Vertex operators and τ-functions*, Proc.Japan Acad., **A57** (1981) 387-392

[41] E.Date, M.Jimbo, M.Kashiwara, T.Miwa, *Transformation group for soliton equations. III. Operator approach to the Kadomtsev-Petviashvili equation*, J.Phys.Soc.Jap., **50** (1981) 3806-3812

[42] E.Date, M.Jimbo, M.Kashiwara, T.Miwa, *Transformation group for soliton equations. IV. A new hierarchy of soliton equations of KP-type*, Physica, **D4** (1982) 343-365

[43] E.Date, M.Jimbo, M.Kashiwara, T.Miwa, *Transformation groups for soliton equations*, in: Proc.RIMS symp. Nonlinear integrable systems — classical theory and quantum theory, M.Jimbo, T.Miwa, eds., World Scientific, Singapore 1983, p.39

[44] E.Date, M.Jimbo, M.Kashiwara, T.Miwa, *Transformation group for soliton equations. V. Quasi-periodic solutions of the orthogonal KP equation*, Publ.RIMS, Kyoto Univ,. **18** (1983) 1111-1119
E.Date, M.Jimbo, M.Kashiwara, T.Miwa, *Transformation group for soliton equations. VI. KP hierarchies of orthogonal and symplectic type*, J.Phys.Soc.Jap., **50** (1981) 3813-3818
M. Jimbo, T. Miwa, *Solitons and Infinite dimensional Lie algebras*, Publ.RIMS, Kyoto Univ,. **19** (1983) 943-1001

[45] K.Ueno, K.Takasaki, *Toda lattice hierarchy*, Adv.Studies in Pure Math., **4** (1984) 1-95

[46] S.Kharchev, A.Marshakov, A.Mironov, A.Morozov, *Generalized Kontsevich Model versus Toda hierarchy and discrete matrix models,* Nucl.Phys., **B397** (1993) 339-378

[47] M.Sato, *Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds*, RIMS Kokyuroku, **439** (1981) 30-40

[48] I.M.Krichever, *Integrating non-linear equations by methods of algebraic geometry*, Funct. Analiz i Prilozh. **11** (1977) 15-31
I.M.Krichever, *Methods of algebraic geometry in theory of non-linear equations*, Sov.Math.Uspekhi, **32** (1977) 183-208
G.Segal, G.Wilson, *Loop groups and equations of KdV type*, Publ.I.H.E.S., **61** (1985) 5-65

[49] V.E.Zakharov, S.V.Manakov, S.P.Novikov, L.P.Pitaevsky, *Theory of solitons. Inverse scattering method*, Moscow, "Nauka", 1980

[50] V.Kac, A.S.Schwarz, *Geometric interpretation of partition function of 2D gravity*, Phys.Lett., **B257** (1991) 329.

[51] S.Kharchev, A.Marshakov, A.Mironov, A.Morozov, A.Zabrodin, *Towards unified theory of 2d gravity,* Nucl.Phys., **B380** (1992) 181-240

[52] S.Kharchev, A.Marshakov, A.Mironov, A.Orlov, A.Zabrodin, *Matrix models among integrable theories: forced hierarchies and operator formalism*, Nucl.Phys., **B366** (1991) 569-601

[53] P.J.Hansen, D.J.Kaup, *Forced non-linear Schrödinger equation*, J.Phys.Soc.Japan, **54** (1985) 4126

[54] M.A.Olshanetsky, A.M.Perelomov, *Classical integrable finite-dimensional systems related to Lie algebras*, Phys.Rept., **71** (1981) 313-400

[55] M.Olshanetsky, A.Perelomov, *Quantum integrable systems related to Lie algebras*, Phys.Rept., **94** (1983) 313-404

[56] O.I.Bogoyavlensky, *On perturbations of the periodic Toda lattice*, Comm. Math. Phys., **51** (1976) 201-209

[57] R.Hirota, *Discrete Two-Dimensional Toda Molecule Equation*, Journ. of Phys.Soc. Japan, **57** (1987) 4285-4288
R.Hirota, Y.Ohta, J.Satsuma, *Wronskian Structures of Solutions for Soliton Equations*, Progr.Theor.Phys.Suppl., **94** (1988) 59-72

[58] A.Leznov, M.Saveliev, *Theory of group, representations and integration of non-linear systems* $x_{a,z\bar{z}} = \exp(kx)_a$, Physica, **3D** (1981) 62-72

[59] A.N.Leznov, M.V.Saveliev, *Theory group methods for integrating non-linear dynamical systems*, Moscow, "Nauka", 1985
O.Babelon,
Universal exchange algebra for Bloch waves and Liouville theory, Comm.Math.Phys., **139** (1991) 619-643

[60] O.Babelon, *Extended conformal algebra and the Yang-Baxter equation*, Phys.Lett., **B215** (1988) 523
J.-L.Gervais, *Gravity-matter couplings from Liouville theory*, Nucl.Phys., **B391** (1993) 287-332

J.-L.Gervais, J. Schnittger, *Continuous spins in 2-D gravity: chiral vertex operators and local fields*, Nucl.Phys., **431** (1994) 273-314

[61] J. Schnittger, *Quantum group structure and local fields in the algebraic approach to 2D gravity*, Theor.Math.Phys., **104** (1995) 158-191

[62] I.Frenkel, N.Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm.Math.Phys., **146** (1992) 1

[63] M.Jimbo, T.Miwa, *Algebraic Analysis of Solvable Lattice Models*, preprint RIMS-981, May 1994

[64] C.Chari, A.Pressley, *A guide to quantum groups*, Cambridge University Press, 1994

[65] C.Fronsdal, A.Galindo, *The Universal T-Matrix,* preprint UCLA/93/TEP/2, January 1993, 16p.

[66] A.Morozov, L.Vinet, *Free-field representation of group element for simple quantum groups,* preprint ITEP-M3/94, CRM-2202; hepth/9409093

[67] D.P.Zhelobenko, *Compact Lie groups and their representations*, Moscow, "Nauka", 1970

[68] L.L.Waksman, Ya.S.Soibelman, *Algebra of functions on the quantum group $SU(2)$*, Funct. Analiz i Prilozh., **23** (1988) 1-14

[69] S.Levendorskii, Y.Soibelman, *Algebra of functions on compact quantum groups, Schubert cells and quantum tori*, Comm.Math.Phys., **139** (1991) 141-170
Y.Soibelman, *Orbit method fort he algebras of functions on quantum groups and coherent states. I*, International Math.Research Notices, **N6** (1993) 151-163

[70] K.Kajiwara, J.Satsuma, *q-Difference version of the two-dimensional Toda lattice equation*, J.Phys.Soc.Jpn., **60** (1991) 3986-3989
K.Kajiwara, Y.Ohta, J.Satsuma, *q-discrete Toda molecule equation*, solv-int/9304001

[71] M.Rosso, *Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra*, Comm.Math.Phys., **117** (1988) 581-593

[72] A.M.Perelomov, *Integrable systems of classical mechanics and Lie algebras*, Moscow, "Nauka", 1990

[73] L.Faddeev, L.Takhtadjan, *Hamiltonian Approach to the Theory of Solitons*, Moscow, "Nauka", 1986

[74] M.Olshanetsky, V.Rogov, *Liouville quantum mechanics on a lattice from geometry of quantum Lorentz group*, Journ.Phys.: Math.Gen., **27** (1994) 4669-4683

[75] I.M.Gelfand, M.I.Graev, I.I Pyatetsky-Shapiro, *Representation theory and automorphic functions*, Moscow, "Nauka", 1966

[76] I.Gradshtein, I.Ryzhik, *Tables of integrals, sums, series and products,* Fizmatgiz, Moscow 1963

[77] Harish-Chandra, *Spherical functions on a semisimple Lie group*, *Amer.Journ.Math.,* **80** (1958) 241-310, 553-613

[78] D.Mamford, *Tata lectures on theta I, II,* Birkhauser, Boston-Basel-Stuttgart, 1984

[79] H.Awata, M.Noumi, S.Odake, *Heisenberg realization for $U_q(SL(n))$ on the flag manifold*, preprint YITP/K-1016, May 1993; hepth/9306010

[80] P.Griffits, F.Adams, *Topics in algebraic geometry and analytic geometry*, Mathematical Notes, **13**, Princeton University Press, 1974

[81] V.G.Drinfeld, *Quantum groups*, Zap.Nauchn.Sem. LOMI, **155** (1986) 19-49
M.Jimbo, *A q-difference analog of $U(\mathcal{G})$ and the Yang-Baxter equation*, Lett.Math.Phys., **10** (1985) 63-69

[82] N.Wallach, *Harmonic Analysis in Homogeneous Spaces*, Dekker, New York, 1973
S.Helgason, *"Groups and Geometric Analysis"*, Academic Press, 1984